Chapter 1

$\mathbb{F}$ ← $\mathbb{C}$: set of all complex number → scalar
Addition and multiplication rules → arithmetic properties ?

vectorspace (set) ← vector ← $\mathbb{F}^n$: set of all lists of n = $x \in \mathbb{F}$ ← List — ordered

list

Define + & M rules

Vector Space / $\mathbb{R}$

data
- S : set
- $+$: $S \times S \to S$
- $\cdot$: $\mathbb{R} \times S \to S$

such that

$$\begin{cases} v + w = w + v \\ c(v + w) = cv + cw \\ (c_1 \cdot c_2) v = c_1 (c_2 w) \end{cases}$$

- Axiom
- Definition
- Proposition / Lemma / Theorem / Corollary

Class 1

Abstract Algebra.

- group < symmetric group S_n. ; linear group GL_n.
- ring $R[x]$ $\mathbb{Z}$ division ✗
- field. $\mathbb{R}$ & $\mathbb{C}$. division ✓.

D. Linear Transformation < D. scalar multiplication. ; D addition

(beginning?)

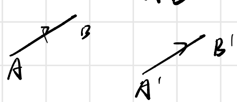
D: identical vector <
- D vector on a plane R^2 ; directed segment — ordered pair of points.
- D translation: move the end without changing length & direction

? how why

(formal)

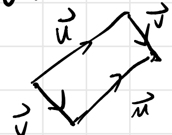
⇒ • D vector is an equivalence class of directed segment modulo translation

$\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are < different as dir. seg $\overrightarrow{AB}$; but same as vector $[\overrightarrow{AB}]$



- D vector addition: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
- commutativity $[\overrightarrow{AB}] + [\overrightarrow{CD}] = [\overrightarrow{CD}] + [\overrightarrow{AB}]$

proof



{ D addition ; parallelogram

- associativity

- distributive laws.

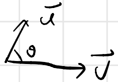
proof: proportional segment + similar triangle.

• D inner product — imposed structure on vector space

: given 2 vectors $\vec{v}, \vec{u}$

we define $\langle \vec{v}, \vec{u} \rangle \in \mathbb{R}$

$$\langle \vec{v}, \vec{u} \rangle = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$$



P: • inner product is defined to be a ~~linear~~ bilinear operation

structure of scalar multiplication & addition < P_1 additivity, P_2 homogeneity

$\Rightarrow$ $\langle \vec{v}_1 + \vec{v}_2, \vec{w} \rangle = \langle \vec{v}_1, \vec{w} \rangle + \langle \vec{v}_2, \vec{w} \rangle$

$\langle \lambda \vec{v}, \vec{w} \rangle = \lambda \langle \vec{v}, \vec{w} \rangle$

$\langle \vec{u}, \vec{u} \rangle = |u|^2$

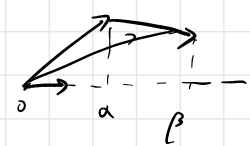
$$\cos\theta_{\vec{u},\vec{v}} = \frac{\langle u, v \rangle}{\langle u, u \rangle^{\frac{1}{2}} \cdot \langle v, v \rangle^{\frac{1}{2}}}$$

$\langle u_1 + u_2, v_1 + v_2 \rangle$

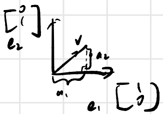
$= \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle + \langle u_1, v_2 \rangle + \langle u_2, v_1 \rangle$

Proof of additivity < $\langle \vec{v}_1 + \vec{v}_2, \vec{1} \rangle = \langle \vec{v}_1, \vec{1} \rangle + \langle \vec{v}_2, \vec{1} \rangle$, homogeneity.

α $\qquad$ β



• D. Basis vector



any other vector in $\mathbb{R}^2$ can be written as linear combination of BV

$V = a_1 e_1 + a_2 e_2$.

• e_1, e_2 forms an orthonormal basis.

orthogonal $\langle e_1, e_2\rangle = 0$; $|e_1| = 1$ $|e_2| = 1$

$\Rightarrow$ claim: $V = \langle V, e_1\rangle e_1 + \langle V, e_2\rangle e_2$.

proof: suppose $V = a_1 e_1 + a_2 e_2$, then we apply $\langle \cdot , e_1\rangle$ to both sides

$$\langle V, e_1\rangle = \langle a_1 e_1 + a_2 e_2, e_1\rangle$$
$$= a_1 \langle e_1, e_1\rangle + a_2 \langle e_2, e_1\rangle$$
$$= a_1 \cdot 1 + a_2 \cdot 0 = a_1$$

• D coordinates

Given a basis E_1, E_2, we say a vector V has coordinates (a_1, a_2), if $V = a_1 E_1 + a_2 E_2$.

Note: change basis $\longrightarrow$ change coordinates.

• "Skewed" basis on $\mathbb{R}^2$

$\downarrow$

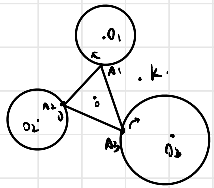
not orthonormal

• In general, if two V E_1, E_2 in $\mathbb{R}^2$ satisfy the property that, any vector $V \in \mathbb{R}^2$ can be written as
$$V = a_1 E_1 + a_2 E_2$$
then we say (E_1, E_2) is a basis.

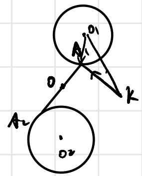
$\Rightarrow$ 2D: non parallel vector definition.

HW 1

6.



$\Leftarrow$ simplified segment case



$\vec{KO} = \frac{1}{3}(\vec{KA_1} + \vec{KA_2} + \vec{KA_3})$

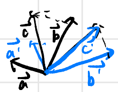
$= \underbrace{\frac{1}{3}(\vec{KO_1} + \vec{KO_2} + \vec{KO_3})}_{\Downarrow\ \vec{u}} + \underbrace{\frac{1}{3}(\vec{O_1A_1} + \vec{O_2A_2} + \vec{O_3A_3})}_{\Downarrow\ \vec{v}}$

for abbreviation, let's see

$\vec{u}$ is clearly a vector w/ fixed length & direction

for $\vec{v}$:

$\because$ for two vectors $\vec{a}$ and $\vec{b}$ with fixed lengths and same angular velocity



geographically, their sum $\vec{c}$ should be a vector with also fixed length and angular velocity

$\therefore$ the addition can be generalized from 2 to n

$\therefore$ $\vec{v}$ is a vector with fixed length and constant angular velocity

same with the segment case

it forms a circle

$\vec{KO} = \frac{1}{2}(\vec{KA_1} + \vec{KA_2})$

$= \frac{1}{2}(\vec{KO_1} + \vec{O_1A} + \vec{KO_2} + \vec{O_2A})$

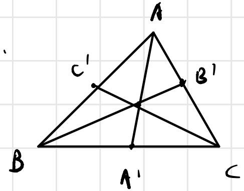
$= \frac{1}{2}(\vec{KO_1} + \vec{KO_2}) + \frac{1}{2}(\vec{O_1A} + \vec{O_2A})$

$\Downarrow$ a vector w/ fixed length & direction

$\Downarrow$ a vector w/ fixed length & direction changing at w

= circle shape movement with angular velocity w

8.



$\because$ Based on Q7: $\overrightarrow{AA'} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$

$\therefore \overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{BC} + \overrightarrow{BA} + \overrightarrow{CA} + \overrightarrow{CB})$

$= 0$

$\therefore$ a triangle can be formed

11. $A(x_a, y_a)$ $B(x_b, y_b)$, C; D E.

$\overrightarrow{AB} = (x_b - x_a, y_b - y_a)$ $M(\frac{x_a + x_b}{2}, \frac{y_a + y_b}{2})$

$\overrightarrow{CD} = (x_d - x_c, y_d - y_c)$ $N(\frac{x_c + x_d}{2}, \frac{y_c + y_d}{2})$ $\Big\rangle Q(\frac{x_a + x_b + x_c + x_d}{4}, \frac{\cdots}{4})$

$\overrightarrow{BC} = (x_c - x_b, y_c - y_b)$ $O(\frac{x_b + x_c}{2}, \frac{y_b + y_c}{2})$

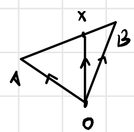
$\overrightarrow{DE} = (x_e - x_d, y_e - y_d)$ $P(\frac{x_d + x_e}{2}, \frac{y_d + y_e}{2})$ $\Big\rangle R(\frac{x_b + x_c + x_d + x_e}{4}, \frac{\cdots}{4})$

$\overrightarrow{AE} = (x_e - x_a, y_e - y_a)$

$\overrightarrow{QR} = (\frac{x_e - x_a}{4}, \frac{y_e - y_a}{4}) = \frac{1}{4}\overrightarrow{AE}$

$\therefore$ parelel

12. ① X lie on AB $\Rightarrow$ $\overrightarrow{OX} = \lambda\overrightarrow{OA} + (1-\lambda)\overrightarrow{OB}$



Suppose $\overrightarrow{AX} = a\overrightarrow{AB}$ $(0 < a < 1)$.

$\because$ X lie on AB

$\therefore \begin{cases} \overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BX} = \overrightarrow{OB} + (1-\lambda)(\overrightarrow{OB} - \overrightarrow{OA}) \\ \overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = \overrightarrow{OA} + \lambda(\overrightarrow{OB} - \overrightarrow{OA}) \end{cases}$

$\therefore \overrightarrow{OX} = a\overrightarrow{OA} + (1-a)\overrightarrow{OB}$

② inverse.

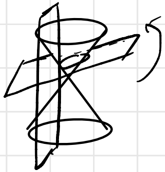
$\overrightarrow{OX} = a(\overrightarrow{OA} - \overrightarrow{OB}) + \overrightarrow{OB}$

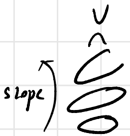
$= a\overrightarrow{BA} + \overrightarrow{OB}$

$\therefore$ X's on AB.

Conic curves.

1. motivation: picture. eigenvalue / product

① 

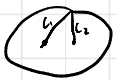


: hyperbola.

: parabola.

: ellipse

: circle.

② 

ellipse = 2 points on R^2, $l_1 + l_2 = L$ (fixed)

dandelin's spheres

2. Definition using equations.

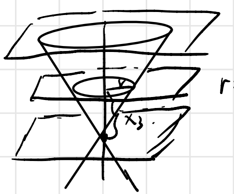
loop. examples : bottom up induction

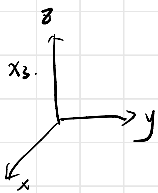
classification : top down deduction

ME, CE.

2.1 ① example.

2.1.1 a cone in R^3: $x_1^2 + x_2^2 = x_3^2$





① can you see it as a linear combination?

2.1.2 a plane in R^3 $a_1x_1 + a_2x_2 + a_3x_3 = a_4$

- a line in R^2: $a_1x_1 + a_2x_2 = a_3$ ② 向量点积

2.1.3 a conic section is given by

$$\left\{ (x_1, x_2, x_3) \;\middle|\; \begin{matrix} x_1^2 + x_2^2 = x_3^2 & ① \\ a_1x_1 + a_2x_2 + a_3x_3 = a_0 & ② \end{matrix} \right\}$$

Solution: if $a_3 \neq 0$, then $x_3 = \dfrac{a_0 - a_1x_1 - a_2x_2}{a_3}$.

plug in x_3 into ①

$$x_1^2 + x_2^2 = \left(\frac{a_0 - a_1x_1 - a_2x_2}{a_3} \right)^2$$

$$\Leftrightarrow Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0$$

if $a_3 = 0$: $a_1x_1 + a_2x_2 = a_0$.

$\{(x_1, x_2, x_3) \mid a_1x_1 + a_2x_2 = a_0\} \Leftrightarrow$ vertical plane

② 2.2. Def

$\Rightarrow$ Def: conic sections are solutions to quadratic eqns:

$$\{ (x_1, x_2) \in R^2 \mid \underbrace{Ax_1^2 + Bx_1x_2 + Cx_2^2}_{\text{quadratic form}} + \underbrace{Dx_1 + Ex_2}_{\text{linear form}} + \underbrace{F}_{\text{constant}} = 0 \}$$

2.2.1 ex:

(1) $x_1^2 + x_2^2 = r^2$ / $\left(\frac{x_1}{r}\right)^2 + \left(\frac{x_2}{r}\right)^2 = 1$: unit circle.

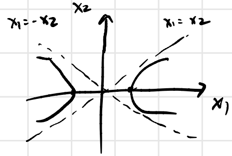
(2) $\left(\frac{x_1}{r_1}\right)^2 + \left(\frac{x_2}{r_2}\right)^2 = 1$

major axis minor axis

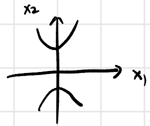
2.2.1

(3) hyperbola:

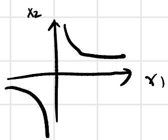
$x_1^2 - x_2^2 = 1$



$-x_1^2 + x_2^2 = 1$

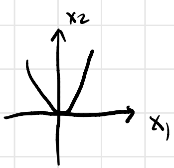


$x_1 x_2 = 1$



4) parabola.

$x_1^2 = x_2$.



2.2.2 elevate examples

$$\left\{\begin{array}{l} x_1^2 + x_2^2 = 1 \\ x_1^2 - x_2^2 = 1 \\ x_2 = x_1^2 \end{array}\right. + \text{affine linear transformation} = \text{all possible conic curves}$$

2.2.2.1 Linear transformation:

Def 1: $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$$

Def 2: $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation

if: (1) $T(c \cdot \vec{v}) = c \cdot T(\vec{v})$, $\forall c \in \mathbb{R}$.

(2) $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$

T (respect) preserve the linear structure on both sides

Prop 1: Fix a basis $\{E_1, E_2\} \in \mathbb{R}$. A linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is determined by how it acts on the basis vectors.

Ex: $T(v) = T(E_1 + E_2)$
$= T(E_1) + T(E_2)$

$T(w) = T(\frac{1}{2} E_1)$
$= \frac{1}{2} T(E_1)$

proof: $\overset{\text{Def of basis } V}{\Rightarrow}$ Any vector $v \in \mathbb{R}^2$ can be written as $V = a_1E_1 + a_2E_2$

$\overset{\text{linearity}}{\Rightarrow}$ $T(v) = T(a_1E_1 + a_2E_2) = T(a_1E_1) + T(a_2E_2) = a_1T(E_1) + a_2T$

Hence, $T(v)$ is determined

Connecting Def 1 and Def 2:

$T(E_1) = a_{11}E_1 + a_{12}E_2$

$T(E_2) = a_{21}E_1 + a_{22}E_2$.

then $T(c_1E_1 + c_2E_2)$

$= c_1a_{11}E_1 + c_1a_{12}E_2 + c_2a_{21}E_1 + c_2a_{22}E_2$

$= (c_1a_{11} + c_2a_{21})E_1 + (c_1a_{12} + c_2a_{22})E_2 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$

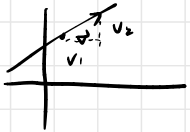
Matrix: $A \cdot B_{ij} = \sum_{k} a_{ik} b_{kj}$

ith row   jth column

$$n\underbrace{\begin{bmatrix} \ A \ \end{bmatrix}}_{L} \cdot L\underbrace{\begin{bmatrix} \ B \ \end{bmatrix}}_{m} = n\underbrace{\begin{bmatrix} AB \end{bmatrix}}_{m}$$

$$(AB)_{ij} = \sum_{k=1}^{L} A_{ik} \cdot B_{kj}$$

1. Line.



$\vec{x}(t) = \vec{x}(0) + t\vec{v}$

$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + t\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

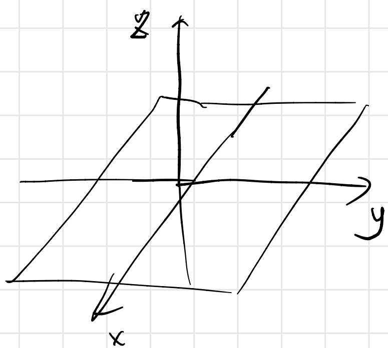
$\begin{cases} x_1(t) = x_1(0) + t v_1 \\ x_2(t) = x_2(0) + t v_2 . \end{cases}$ $\begin{cases} v_2 x_1(t) = v_2 x_1(0) + t v_1 v_2 \\ v_1 x_2(t) = v_1 x_2(0) + t v_1 v_2 \end{cases}$

$\Rightarrow \quad v_2 x_1(t) - v_1 x_2(t) = v_2 x_1(0) - v_1 x_2(0)$

$x_1 \mapsto x$
$x_2 \mapsto y$ $\quad ax + by = c.$
$v_2 = a$
$-v_1 = b$

$ax_1 + bx_2 = c = y = f(x_1, x_2) \in \mathbb{R}^3$

$ax_1 + bx_2 = f(x_1, x_2) = c.$



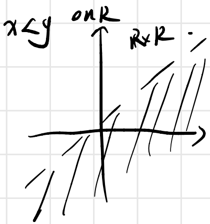
~~level~~
~~level curve~~
~~level~~ set. Given $f(\vec{x})$, the level sets of f are the sets:
$\{ f(\vec{x}) = c : \vec{x} \in \mathbb{R}^n \}$

They are also projections: e.g.

* Equivalent class:

Relation on a set X is a set $R \subseteq X \times X$ st $\{ xRy \text{ is true} \}$

$x \le y$ on $\mathbb{R}$



D Equivalence relation:
(1) reflexive $\quad xRx$ is true.
(2) symmetry $\quad xRy \Rightarrow yRx$
(3) Transitivity $\quad xRy + yRz \Rightarrow xRz$

E.C.
$\downarrow$
$x \sim y \quad xRy$

e.g. $x \sim y$ iff $x - y$ is a multiple of 3

$\Rightarrow [0], [1], [2].$

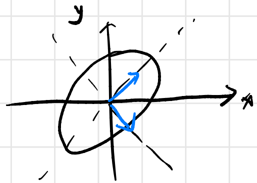
disjoint union: $X = \bigsqcup_{[x]} [x]$

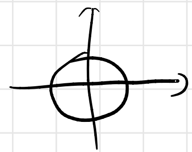
$(X, \sim)$

2. additional conic section:

2.1 $a x_1^2 + 2b x_1 x_2 + c x_2^2 = A X_1^2 + B X_2^2$

e.g. $x^2 - xy + y^2 = 1 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2/3} = 1$





Step 1: find two basis vectors on $y = \pm x$

$x \mapsto \frac{(x+y)}{\sqrt{2}} \iff \begin{pmatrix}1\\0\end{pmatrix} = \left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right]\frac{1}{\sqrt{2}}$

$y \mapsto \left(\frac{x-y}{\sqrt{2}}\right) \quad \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\-1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}$

Step 2: replacement:

$\frac{(x+y)^2}{2} - \frac{(x+y)(x-y)}{2} + \frac{(x-y)^2}{2} = 1$

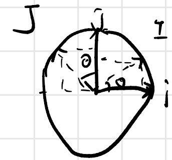
$\frac{X^2}{2} + \frac{Y^2}{2/3} = 1$

Orthogonal diagonalization.

Rotation Lin. $T: R^2 \to R^2$

$[i, j]\begin{bmatrix}x\\y\end{bmatrix} \to [i, j]\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

Change of coordinate in rotation.



1. Change of basis

$[I, J] = [i, j]\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ $\quad I = \cos\theta\, i + \sin\theta\, j$, $\quad J = -\sin\theta\, i + \cos\theta\, j$

2. Change of coordinates

$x = x\hat{i} + y\hat{j} = [\hat{i}, \hat{j}]\begin{bmatrix}x\\y\end{bmatrix}$

$x = XI + YJ = [I, J]\begin{bmatrix}X\\Y\end{bmatrix}$

$= [i, j]\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}$

$x^{\hat{i}\hat{j}} = (\hat{i}\hat{j} \to IJ)\, x^{IJ}$

$= XI + YJ = (X\cos\theta - Y\sin\theta)\, i + (X\sin\theta + Y\cos\theta)\, j$

$x = X\cos\theta - Y\sin\theta \quad y = X\sin\theta - Y\cos\theta$.

$ax^2 + bxy + cy^2 \iff AX^2 + BXY + CY^2$

Review:

1. Axiom

1.1.1 geometric def of a vector:

- directed segment
- vectors: dir seg upto translation

1.1.2 scalar multiplication & addition

1.1.3 inner product on $\mathbb{R}^2$: $\langle \vec{u}, \vec{v} \rangle = |\vec{v}| \cdot |\vec{u}| \cos\theta$

× based on setting up coordinate system

1.2.1 basis and coordinates

$\{E_1, E_2\}$ is a basis, iff for any $V \in \mathbb{R}^2$ there exists a unique way to write $V = C_1E_1 + C_2E_2$

(C_1, C_2) are coordinates of V with respect to $\{E_1, E_2\}$

1.2.2. Linear transformation

$T: \mathbb{R}^2 \to \mathbb{R}^2$

Def 1: LT is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ that respects "linear structure" $\Rightarrow$ s.t. m/A

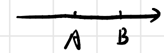
Prop1: $T(\vec{0}) = \vec{0}$

Prop2: $T(C_1E_1 + C_2E_2) = C_1T(E_1) + C_2T(E_2)$

hence, if we know $T(E_1)$, $T(E_2)$, we know $T(v)$ for any $v \in \mathbb{R}^2$

Prop3: rotation $\in$ LT; Reflection over a line $\in$ LT } ① preserve length & angles between vectors

1.2.3. Affine LT

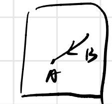
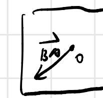
eg. 

$A + B = ?$ × well defined

$A - B = ?$ ✓ $\vec{BA}$

terminology:

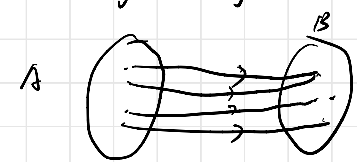
- vector space = linear space } forget the origin } choose ~
- affine space = affine linear space

- A^2: 2-D affine space. 
- $\mathbb{R}^2$: ~ linear space 

$$\boxed{\cancel{A} \bullet \ \text{"}-\text{"}: A^2 \times A^2 \to \mathbb{R}^2, \quad (A, B) \mapsto A - B = \vec{BA}}$$

Pre. Axiom

- If A, B are sets, a map $f: A \to B$ is an assignments of elements in B to A



- Given 2 sets A, B, we can define "Cartesian product"

$A \times B = \{(a, b) \mid a \in A, b \in B\}$

Ex: $A = \{1, 2, 3\}$

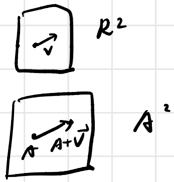
$B = \{e, f\}$

$A \times B = \{(1,e), (1,f), (2,e), (2,f), (3,e), (3,f)\}$

type declaration.

"+" : $A^2 \times R^2 \to A^2$

implementation $(A, \vec{v}) \mapsto A + \vec{v}$



Def. Affine linear transformation:

$A^2 \to A^2$

$(x_1, x_2) \mapsto (\tilde{x}_1, \tilde{x}_2)$ $\quad \tilde{x}_1 = a_{11}x_1 + a_{12}x_2 + a_{10}$

$\tilde{x}_2 = a_{21}x_1 + a_{22}x_2 + a_{20}$

move dots, not vectors. $\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix}$

2. Def of Quadratic form.

$Q(\vec{x}) = ax_1^2 + bx_1x_2 + cx_2^2$. homogeneous degree 2 polynomial in x_1, x_2 variables

$\vec{x} = (x_1, x_2)$.

3. Propositions.

3-1 $\Rightarrow$ Solve $Q(\vec{x}) = 1$

solutions: $\begin{cases} \cdot \text{ ellipse} \\ \cdot \text{ hyperbola.} \\ \cdot \text{ empty curve} \end{cases}$

3.2. Any quadratic form

$Q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$

can be written in a new coordinate $(\tilde{x}_1, \tilde{x}_2)$

as one of the follow form

① $\tilde{x}_1^2 + \tilde{x}_2^2$ $\qquad$ ①' $-\tilde{x}_1^2 - \tilde{x}_2^2$

② $\tilde{x}_1^2 - \tilde{x}_2^2$

③ $\tilde{x}_1^2$

④ 0.

where. $\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for some a_{ij}. that

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$

Proof: Suppose $a \neq 0$

$$ax_1^2 + bx_1x_2 + cx_2^2$$
$$= a\left[x_1^2 + \frac{b}{a}x_1x_2\right] + cx_2^2$$
$$= a\left[x_1^2 + 2x_1 \cdot \frac{b}{2a}x_2 + \left(\frac{b}{2a}x_2\right)^2 - \left(\frac{b}{2a}x_2\right)^2\right] + cx_2^2$$
$$= a\left[x_1 + \frac{b}{2a}x_2\right]^2 - \left(c - \frac{b^2}{4a}\right)x_2^2$$

① if $a>0$, $c > \frac{b^2}{4a}$, then define

$$\tilde{x}_1 = \sqrt{a}\cdot\left(x_1 + \frac{b}{2a}x_2\right)$$
$$\hat{x}_2 = \sqrt{c - \frac{b^2}{4a}}\cdot x_2$$
$$Q = \tilde{x}_1^2 + \tilde{x}_2^2$$

② if $a<0$, $c < \frac{b^2}{4a}$

$$Q = -\tilde{x}_1^2 - \hat{x}_2^2$$

$$\tilde{x}_1 = \sqrt{|a|}\left(x_1 + \frac{b}{2a}x_2\right)$$
$$\hat{x}_2 = \sqrt{\left|c - \frac{b^2}{4a}\right|}\, x_2^2$$

e.g. $Q = x^2 + y^2 + 2x$
$$= (x+1)^2 + y^2 - 1$$
$$\tilde{x} = x+1$$
$$\tilde{y} = y$$

Prop: 3.2 Any quadratic form

$$Q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 + d\cdot x_1 + e\cdot x_2 + f.$$

can be written in a new affine linear coordinate $(\tilde{x}_1, \tilde{x}_2)$

as one of the follow form

① $\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{C}$ ①' $-\tilde{x}_1^2 - \tilde{x}_2^2 + \tilde{C}$

② $\tilde{x}_1^2 - \tilde{x}_2^2 + \tilde{C}$

③ $\tilde{x}_1^2 + \tilde{C}$

④ $\tilde{C}$

1. Principal Axis Theorem:

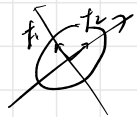
$\Uparrow$ Any QF in 2 variables has two perpendicular principal axes

2. Orthognal Diagolization Thm:

Any QF can be expressed as $Ax_1^2 + Bx_2^2$ in a suitable coordinate system

ODT implies ($\Rightarrow$) PAT

PAT $\Rightarrow$ ODT:



$x_1 \to -x_1$

$x_2 \to -x_2$

$\Rightarrow Ax_1^2 + 2Bx_1x_2 + Cx_2^2$

$= Ax_1^2 - 2Bx_1x_2 + Cx_2^2$

$\therefore$ $2Bx_1x_2 = 0$

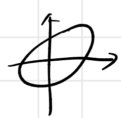
Corollary: Any QF can be expressed in the "normal form"

$x_1^2 + x_2^2$, $x_1^2 - x_2^2$, $-x_1^2 - x_2^2$, x_1^2, x_2^2, 0

$ax_1^2 + \cdots + f = 0$

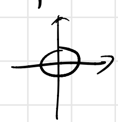
e.g. $x^2 - xy + y^2 = 1$

$\left(\frac{x+y}{\sqrt{2}}\right)^2 + \left(\frac{x-y}{\sqrt{2}}\right)^2 = 1$



$x = \binom{1}{0}$

$y = \binom{0}{1}$



$x' = \binom{1}{1}$

$y' = \binom{1}{-1}$

more on:

Linear Transformation

$\in$ orthognal transformation

prop: $\langle \vec{x}, \vec{y} \rangle = \langle T\vec{x}, T\vec{y} \rangle$

$|\vec{x}| \cdot |\vec{y}| \cdot \cos\theta = |T\vec{x}| \cdot |T\vec{y}| \cdot \cos\theta$

eg 1. Reflection

$\binom{dx}{y} = D\binom{x}{y} \iff D = \begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix}$

$D' = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ dy \end{bmatrix}$

$D'' = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

Eg. 2. counterclockwise rotation.

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

LT:

$$(T\vec{E_1}, T\vec{E_2}) = (\vec{E_1}, \vec{E_2}) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (\vec{E_1}a_{11} + \vec{E_2}a_{21}, \; \vec{E_1}a_{12} + \vec{E_2}a_{22})$$

1×2 $\quad 2\times 2$

$$\vec{v} = c_1\vec{E_1} + c_2\vec{E_2}$$
$$= (\vec{E_1}, \vec{E_2}) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$T(\vec{v}) = (T\vec{E_1}, T\vec{E_2}) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= (\vec{E_1}, \vec{E_2}) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

1×2 $\quad 2\times 2$ $\quad 2\times 1$

$$= (\vec{E_1}, \vec{E_2}) \begin{pmatrix} \check{c}_1 \\ \check{c}_2 \end{pmatrix}$$

$\check{c}_1 = a_{11}c_1 + a_{12}c_2$

$\check{c}_2 = a_{21}c_1 + a_{22}c_2$

→ coordinates for $T\vec{v}$

need 2 know vectors to inversely calculate T

Ex (1). Stretching:

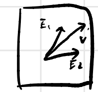
$T(E_1) = 2\vec{E_1}$

$T(\vec{E_2}) = \frac{1}{3}\vec{E_2}$

Projection (2)

$T(E_1) = 0$

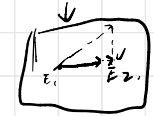
$T(E_2) = \vec{E_2}$.



$T(\vec{v}) = T(aE_1 + bE_2)$

$= bT(E_2)$

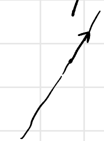
$= b\cdot\vec{E_2}$



projection2: (3) $T(E_1) = \frac{1}{2}E_2$

$T(E_2) = E_2$.

image of R^2 under T is the subspace $R\cdot\vec{E_2}$



Operations on LT (in $\mathbb{R}^2 \to \mathbb{R}^2$)

1. Addition & scalar multiplication: $T_1 : V \to W$, $T_2 : V \to W$ are LT

Then $c_1T_1 + c_2T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ preserves LT

Def: $(c_1T_1 + c_2T_2)(\vec{v})$
$= c_1T_1(\vec{v}) + c_2T_2(\vec{v})$

If $[T_1] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $[T_2] = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$[c_1T_1 + c_2T_2] = c_1[T_1] + c_2[T_2]$

2. composition

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "T_1"] \arrow[rr, bend right, "T_2 \circ T_1"'] & \mathbb{R}^2 \arrow[r, "T_2"] & \mathbb{R}^2 \end{tikzcd}$$

$\vec{v} \mapsto T_1(\vec{v}) \mapsto T_2(T_1(\vec{v}))$

$T_2 \circ T_1 = [T_2] \cdot [T_1]$

Ex: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} \right)$

$= \begin{pmatrix} 0 & \frac{1}{3} \\ 2 & 0 \end{pmatrix}$

Develop matrix multiplication skills

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

< column / row

Properties:

(1) $[T_1] \circ [T_2] \neq [T_2] \circ [T_1]$

(2) $([T_1] \cdot [T_2]) \cdot [T_3] = [T_1]([T_2] \cdot [T_3])$ associativity

(3) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ identity matrix $[I] \cdot [T] = [T]$

A map $f: A \to B$ is invertible iff f is bijective $\begin{cases} \text{injective} \\ \text{surjective} \end{cases}$

A. linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is invertible iff there exists $\exists$ a $S: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $S \circ T = Id$

Zero/ projection matrix is invertible

⇑

lose too much information

lemma: $[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Then $[T]$ is invertible, iff

$\det([T]) \neq 0$
$\shortparallel$
$ad - bc$

$[T] \cdot [T]^{-1} = [Id]$

$[T]^{-1} \cdot [T] = [Id]$

$[T]^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

* If $ad = bc$ then e_1 e_2 $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Ae_1 Ae_2 $\begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix}$ will become proportional

$\frac{a}{b} = \frac{c}{d}$

determinant:

areas of parallelogram formed by basis vectors

orthognal transformation: presere $\left\{ \begin{array}{l} \text{inner product } \langle T\vec{x}_1, T\vec{x}_2\rangle = \langle \vec{x}, \vec{y}\rangle \\ \text{length} \\ \text{angle} \end{array} \right.$

Axiom:

- let R^2 be equipped with the standard inner product

  $\langle \vec{u}, \vec{w}\rangle = |\vec{u}|\cdot|\vec{v}|\cdot \cos\theta$

- A basis $\{\vec{E}_1, \vec{E}_2\}$ is orthonormal if $\begin{array}{l} |\vec{E}_i| = 1 \\ \langle \vec{E}_1, \vec{E}_2\rangle = 0 \end{array}$

- $\langle \vec{v}, \vec{w}\rangle = v_1w_1 + v_2w_2 = (w_1\ w_2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (v_1\ v_2)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \vec{w}^t \cdot \vec{v} = \vec{v}^t \cdot w$

  - $[A]$ matrix of size $m \times n$.

    $[A]^t_{ij} = [A]_{ji}$ e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

    $1 \le i \le n$ $\quad 1 \le j \le m$

Def: A linear transformation $T: R^2 \to R^2$ is orthogonal if for any $\vec{v}, \vec{w} \in R^2$, $\langle T\vec{v}, T\vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle$

prop: let $\{\vec{e}_1, \vec{e}_2\}$ be an ONB of R^2

let $T: R^2 \to R^2$ be a linear trans and $[T]$ the matrix form w.r.t $\{\vec{e}_1, \vec{e}_2\}$

then T is an orthognal trans iff $[T]^{-1} = [T]^t$.

Ex: rotation

$$[T] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$[T]^t = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Q: 1.

$$\frac{x^2}{4} + y^2 = 1 \Rightarrow x'^2 + y'^2 = 1$$

$$x' = \frac{x}{2} \qquad \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$y' = y$$

+ rotation

Q_2:

rotate at diffent angle.

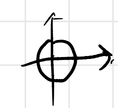
Prove that: any OT is either reflection or rotation

$\uparrow e_2$

$\rightarrow e_1$

$\Rightarrow$ $T(e_1) = \cos\theta e_1 + \sin\theta e_2$

e_2 $\quad Te_1$ $\quad Te_2$ falls on the unit circle



1. orthogonal transformation

1.1 Axioms

- vector space : $\mathbb{R}^2$
- inner product : $\langle -,- \rangle$ $\begin{cases} \cdot \text{ linear in both slot} \\ \cdot \ \langle u,u \rangle \geq 0 \\ \cdot \text{ symmetric } \langle u,v \rangle = \langle v,u \rangle \ \forall u,v \in \mathbb{R}^2 . \end{cases}$
- $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an orthogonal trans of $\mathbb{R}^2$
  if $\forall u,v \in \mathbb{R}^2$, $\langle u,v \rangle = \langle Tu, Tv \rangle$

① • prop: if $T: \mathbb{R}^2 \to \mathbb{R}^2$ preserves length. i.e. $|T\vec{u}| = |\vec{u}|$ $\forall u,v \in \mathbb{R}^2$
then $\langle u,v \rangle = T\langle u,v \rangle$

Pf: By "polarization"

$\Rightarrow \langle u+v, u+v \rangle = \langle T(u+v), T(u+v) \rangle$

$\langle u,u \rangle + \langle v,v \rangle + \underline{2\langle u,v \rangle} = \langle Tu, Tu \rangle + \langle Tv, Tv \rangle$
$\underline{+2\langle Tu, Tv \rangle}$

$\langle u,v \rangle = \langle Tu, Tv \rangle$

1.2. Def see previous

1.3 properties

① Given an orthonormal basis $(\vec{e_1}, \vec{e_2})$ then
an orthonormal $T \in O(\mathbb{R}^2)$, can be presented
as $[T] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

so that, given a vector $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$,

$$T\vec{v} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

② prop: $[T]^{-1} = [T]^t$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Pf: For any 2 vectors $\vec{v}, \vec{w} \in \mathbb{R}^2$ $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$\langle \vec{v}, \vec{w} \rangle = (v_1, v_2) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = (v_1, v_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$\langle T\vec{v}, T\vec{w} \rangle = ([T]\vec{v})^t \cdot ([T] \cdot \vec{w})$

By axiom: $(AB)^T = B^T A^T$

$= \vec{v}^t \cdot [T]^t \cdot [T] \cdot \vec{w}$

lemma: Let M be a 2x2 matrix, if for any $\vec{v}, \vec{w} \in \mathbb{R}^2$,
we have $\vec{v}^t \cdot M \cdot \vec{w} = 0$ then $M = 0$

$M = 0 \iff \forall v, w \in \mathbb{R}^2, v^t \cdot M \cdot w = 0$

Pf: $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ if M is not zero.

$\Rightarrow$ at least one entry of M is non zero

say $m_{11} \neq 0$

let $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$0 = \vec{v}^t \cdot M \cdot \vec{w} = (1, 0) \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= (1, 0) \begin{pmatrix} a_{11} \\ m_{21} \end{pmatrix}$

$= m_{11}$

In fact $m_{ij} = e_i^t \cdot M \cdot e_j$. Thus $m_{ij} = 0 \ \forall ij$

Since $0 = \langle \vec{v}, \vec{w} \rangle - \langle T\vec{v}, T\vec{w} \rangle$

$= \vec{v}^t \cdot I_2 \cdot \vec{w} - \vec{v}^t \cdot [T]^t \cdot [T] \vec{w}$

$= \vec{v}^t \cdot (I_2 - [T]^t \cdot [T]) \cdot \vec{w}$

By the previous lemma, we have $I_2 - [T]^t \cdot [T] = 0$

rotation:

2. $\det R(\theta) = \cos^2\theta - (\sin\theta)(-\sin\theta) = 1$

A linear transformation preserves orientation iff $\det > 0$

2.1 reflection:

$\det \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1$

Axiom

for any $T \in O(\mathbb{R}^2)$

$\det([T]) = \pm 1$

2.2. Special orthogonal transformation.

$SO(\mathbb{R}^2) \subseteq O(\mathbb{R}^2)$

$\det A = \pm 1 \qquad \det A = +1$

proof: $1 = \det(I_2) = \det([T]^t \cdot [T])$

$= \det([T]) \cdot \det([T])$

$\leftarrow$ Axiom:

$\det(A \cdot B)$

$= \det A \cdot \det B$

$\det A = \det A^t$

Complex number

Axiom: ① i or $\sqrt{-1}$ as one of the solution to $x^2+1=0$

2 solutions: $\sqrt{-1}$, $-\sqrt{-1}$

Def: $z = a+bi$, $a, b \in \mathbb{R}$

$Re(z) = a$ $Im(z) = b$

Prop: 1. add & multi rules.

2.1 conjugation:

$z = a+bi$

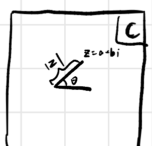
$\bar{z} = a + b(-i)$ i to $-i$

2.2. if we are given a deg 2 polynomial with real coeff

$f(x) = x^2 + c_1 x + c_0$, $c_1, c_0 \in \mathbb{R}$

Then the equation $f(x)=0$ sometimes has 2 complex solutions,

they are in complex conjugate.



$|z| = \sqrt{a^2+b^2}$

$\theta = \arg(z)$ $(\theta = \theta + 2\pi)$

polar form: $z = r \cdot e^{i\theta}$

①

$= r \cdot \cos\theta + i \cdot r\sin\theta$

proof: $e^{i\theta} = \cos\theta + i\sin\theta$

taylor expansion on both sides

② $z_1 z_2 = (r_1 e^{i\theta_1}) \cdot (r_2 e^{i\theta_2})$

$= (r_1 r_2) e^{i(\theta_1+\theta_2)}$

③ $\bar{z} = r \cdot e^{-i\theta}$

④ $i = e^{\frac{\pi}{2} i}$

multiplication by i rotates 90°

3. Fundamental Theorem of Algebra:

let $f(z) = z^n + c_{n-1} \cdot z^{n-1} + c_{n-2} \cdot z^{n-2} + \cdots + c_0$

be a degree n polynomial where $c_i \in \mathbb{C}$

Then $f(z)=0$ has n solutions (counted with multiplicities)

- multiplicity: $f(z) = (z-z_1)^{m_1} \cdots (z-z_r)^{m_r}$, $m_i \geq 1$ integers

z_i are distinct

$m_1 + \cdots + m_r = n$

$z = a+bi$

$= r \cdot e^{i\theta}$

$\sqrt{a^2+b^2} = \sqrt{z \cdot \bar{z}} = |z|$ $\cos\theta + i\sin\theta$

$\Rightarrow |z \cdot w| = \sqrt{zw \cdot \overline{zw}} = \sqrt{z\bar{z}w\bar{w}} = |z| \cdot |w|$

$|e^{i\theta}| = 1$

$u = e^{i\theta} = \cos\theta + i\sin\theta$ $w = x+yi$

$u \cdot w = (\cos\theta + i\sin\theta)(x+yi)$

$= \cos\theta x - \sin\theta y + i(\sin\theta x + \cos\theta y)$

$= X + Yi$

$w \mapsto uw$ is rotation by θ of u.

More generally.

$z \cdot w = |z| \cdot |w| \cdot e^{i(\theta_1+\theta_2)}$

$z^n = r^n \cdot e^{in\theta}$

nth roots of unity.

$z = r(\cos\theta + i\sin\theta)$ satisfies $z^n = 1$

$\Leftrightarrow$ $r^n = 1$ $(r=1)$ $\quad n\theta = 2\pi \cdot k \quad k = 0, \pm 1, \pm 2 \ldots$

$\Rightarrow$ $z = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$. $\quad k = 0, 1, \ldots, n-1$

ex. $n = 3$

$\cos 3\theta + i\sin 3\theta = \cos\theta + i\sin\theta$

$\theta = \frac{2\pi \cdot k}{3}$

$\theta_1 = 0. \qquad z = 1$

$\theta_2 = \frac{2\pi}{3} \qquad z = \cos\frac{2\pi}{3} + i\sin\frac{2}{3}\pi.$

$\theta_3 = \frac{4\pi}{3} \qquad z = \cos\frac{2\pi}{3} - i\sin\frac{2}{3}\pi.$

Thm. Complex roots come in conjugaton

Pf: $P(z) = z^n + a_1 z^{n-1} + \cdots a_n = (z - z_1)^{m_1} \cdots (z - z_r)^{m_r}$

$P(\bar{z}) = \overline{P(z)} = \bar{z}^n + \bar{a}_1 \bar{z}^{n-1} + \cdots \bar{a}_n. = (\bar{z} - \bar{z}_1)^{m_1} \cdots (\bar{z} - \bar{z}_r)^{m_r}$

$z_1 + \cdots z_n = -a_1 \qquad \Rightarrow \qquad z^2 + pz + q = 0.$

$z_1 \cdots z_n = (-1)^n a_n \qquad\qquad z_+ + z_- = -p$

$z_+ \cdot z_- = q$

Similarity transformations

$\vec{y} = T\vec{x}$

$\vec{x} = A\vec{x}'$

$\vec{y} = A\vec{y}'$

$T\vec{x} = A\vec{y}'$

$A^{-1}T\vec{x} = \vec{y}'$

$A^{-1}TA\vec{x}' = \vec{y}'$

$A^{-1}TA$ is the matrix of the

T that sends $\vec{x}' \mapsto \vec{y}'$

T " $\vec{x} \mapsto \vec{y}$ "

$A^{-1}TA$ and T are similar

"related by a similarity transformation".

1. Classification Theorems

Def 1) describe the types of allowed T

2) pick a normal (canonical) form, which is a representative for each class

3) a classification theorem proving that each object in X is equivalent to exactly one representative

Ex 1: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow$ prototype $x^2+y^2=1$

$xy=1$

$y=x^2$

∵ LA can be used to study/classify systems of linear & quadratic eqn.s.

$y_1 = a_{11}x_1+\cdots+a_{1n}x_n$

$\vdots$

$y_m = a_{m1}x_1+\cdots a_{mn}x_n$

$Q = q_{11}x^2+2q_{12}x_1x_2+\cdots q_{nn}x_n^2$

2. The Rank Theorem

$y_i = b_{i1}\bar{y}_1 + b_{i2}\bar{y}_2+\cdots+b_{im}\bar{y}_m$

$x_i = c_{j1}\bar{x}_1 + c_{j2}\bar{x}_2+\cdots+c_{jn}\bar{x}_n$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$\downarrow$

$$\begin{bmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 & \vdots \\ 0 & \cdots & & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

r = rank of the system

3. The Inertia Theorem

Q can be expressed as

$$x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

$1 \leq p+q \leq n$

p, q = inertia coefficients.

(eliminate cross terms)

1.4 4 Theorems of LA.

1.4.1. Motivation: Classification

- Transformation (Group: invertible): G.
- Def equivalence after transformation

$$G \curvearrowright X$$

action

$$a: G \times X \to X$$

elements in G & X

Two elements $x, y \in X$ are equivalent, if there exists $g \in G$, such that $g \cdot x = y \iff x = g^{-1} \cdot y$

we denote $x \sim y$

- For a system of linear eq.

$$\begin{cases} y_1 = a_{11}x_1 + \cdots + a_{1n}x_n \\ y_2 = \\ \vdots \\ y_m = a_{m1}x_1 + \cdots + a_{mn}x_n \end{cases}$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

- describe the equivalence class

$$X/\sim$$

① an equivalence class can be described as a model element

e.g. $\{\text{conic curves}\}$

$= \{\text{ellipses}\} \cup \{\text{parabola}\} \cup \{\text{hyperbola}\}$

$$\bigcirc \qquad y^2 = x \ \cup \qquad x^2 - y^2 = 1$$

1.4.2. Rank Theorem:

Premise: rank:

rank measures number of linearly independent columns.

Ex: $T: \mathbb{R} \to \mathbb{R}^2$

$$t \mapsto (2t, 3t)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot (x_1)$$

$$\begin{cases} Y_1 = 2y_1 \\ Y_2 = 3y_1 - 2y_2 \\ X_1 = x \end{cases}$$

$$\begin{cases} Y_1 = x_1 \\ Y_2 = 0. \end{cases}$$

Def: Any linear eqn of the form

$\vec{y} = A\vec{x}$ where $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$ $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

can be simplified by a "change of variable"

$\vec{y} = B\vec{Y}$, $\vec{x} = C \cdot \vec{X}$

B is $m \times m$ invertible C is $n \times n$ invertible.

so that

$$\vec{Y} = \underbrace{\left(\begin{array}{ccc|c} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ \hline & 0 & & 0 \end{array}\right)}_{\Lambda}\vec{X}$$

where

$$\begin{aligned} Y_1 &= X_1 \\ Y_2 &= X_2 \\ &\vdots \\ Y_r &= X_r \\ Y_{r+1} &= 0 \\ &\vdots \\ Y_m &= 0 \end{aligned}$$

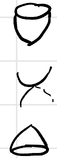
1.4.3 Inertia Theorem

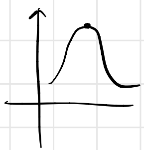
1.4.3.1 motivation

graph of functions: $f_1(x,y) = x^2+y^2$

$f_2(x,y) = x^2-y^2$

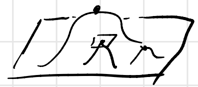
$f_3(x,y) = -x^2-y^2$





$\Rightarrow$

quadratic



1.4.3.2 Def.

Given a quadratic form

$$Q(\vec{x}) = \vec{x}^{\,t} \cdot M \cdot \vec{x}$$

$M \downarrow \begin{pmatrix} a_{11} \cdots a_{1n} \\ a_{n1} \cdots a_{nn} \end{pmatrix}$ symmetric, $a_{ij} = a_{ji}$

$\vec{x} \downarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

e.g.

$$a x_1^2 + b x_1 x_2 + c x_2^2$$

$$= (x_1, x_2) \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (x_1, x_2) \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (x_1, x_2) \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \star$$

up to linear transformation

$\vec{x} = A \vec{X}$, $A \in GL(\mathbb{R}^n)$ $n \times n$ invertible matrices

$$Q(\vec{x}) = \vec{x}^{\,t} \cdot M \cdot \vec{x}$$
$$= (A\vec{X})^t \cdot M \cdot A \cdot \vec{X}$$
$$= \vec{X}^t \cdot A^t \cdot M \cdot A \cdot \vec{X}$$
$$= \vec{X}^t \cdot \Lambda \cdot \vec{X}$$

$$\Lambda = \begin{pmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{p} & & \\ & \overbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}^{q} & \\ & & \overbrace{\begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix}}^{n-p-q} \end{pmatrix}$$

Comparison

1.4.1 rank: input: $\vec{y} = A\vec{x}$

want to find $\vec{y} = B\vec{Y}$

$\vec{x} = C\vec{X}$

$B\vec{Y} = A \cdot C\vec{X}$

$\vec{Y} = B^{-1} A \cdot C\vec{X}$

rank easier cuz can choose B & C

inertia can only choose A

1.4.4. Orthogonal Diagonalization Theorem

Def: Given $Q(\vec{x}) = \vec{x}^t \cdot M \cdot \vec{x}$

Want to find $A \in O(\mathbb{R}^n)$ ← orthogonal group

$= \{ M \in GL(n) \mid M^t \cdot M = I \}$

$= \{ M : \mathbb{R}^n \to \mathbb{R}^n \mid \langle Mu, Mv \rangle = \langle u, v \rangle \ \forall u, v \in \mathbb{R}^n \}$

s.t. if we set $\vec{x} = A \cdot \vec{X}$

then $\vec{x}^t \cdot M \cdot \vec{x} = \vec{X}^t \underbrace{(A^t \cdot M \cdot A)}_{\Lambda} \vec{X}$

$$\Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_n \end{pmatrix} \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \lambda_n$$

relatable to Inertia Th.

model in eq class is also affected by operations allowed.

1.4.5 Jordan Normal Form:

Def: Given $T: \mathbb{R}^n \to \mathbb{R}^n$

$\vec{y} = A \cdot \vec{x}$

find LT/ change of basis of $\mathbb{R}^n$

$\vec{x} = B \vec{X}$

$\vec{y} = B \vec{Y}$ $\quad B \in GL(n)$

s.t. the equation in $\vec{X}$, $\vec{Y}$, simplifies

$B \cdot \vec{Y} = A \cdot B \cdot \vec{X}$

$\vec{Y} = \underbrace{B^{-1} \cdot A \cdot B}_{= \Lambda} \vec{X}$

$$\begin{array}{c} r_1 \\ r_2 \\ \\ \end{array} \begin{pmatrix} \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix} & 0 \\ 0 & 0 & \ddots \end{pmatrix}$$

Λ consists of block diagonal matrices

each block is $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$ ← Jordan block

More on Jordan Canonical.

Given: $\dot{x}_1 = a_{11}x_1 + \cdots + a_{1n}x_n$

$\vdots$

$\dot{x}_n = a_{n1}x_1 + \cdots + a_{nn}x_n$

Allowed: $x_1 = c_{11}\bar{x}_1 + \cdots + c_{1n}\bar{x}_n$

$\vdots$

$x_n = c_{n1}\bar{x}_1 + \cdots \; c_{nn}\bar{x}_n$

$a_{ij}, c_{ij} \in \mathbb{C}$

Example: Single m^{th} order linear ODE

$$\left(\frac{d}{dt} - \lambda\right)^m y = 0 \quad \lambda \in \mathbb{C}$$

Set $y = x_1$, $\left(\frac{d}{dt} - \lambda\right) y = x_2$, $\left(\frac{d}{dt} - \lambda\right)^2 y = x_3, \cdots \left(\frac{d}{dt} - \lambda\right)^{m-1} y = x_m$

$$\begin{cases} \dot{x}_1 = \lambda x_1 + x_2 \\ \dot{x}_2 = \lambda x_2 + x_3 \\ \vdots \\ \dot{x}_{m-1} = \cdots \lambda x_{m-1} + x_m \\ \dot{x}_m = \lambda x_m \end{cases}$$

$\Leftrightarrow$ Jordan cell of size m w/ eigenvalue λ.

Thm 1: Every constant coefficient system of n linear 1st order ODEs in n unknowns can be transformed by a complex linear change of coords to exactly one of the Jordan systems of $m_1 + \cdots + m_k = n$

$$J_\lambda = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \leftarrow \text{Jordan Block}$$

A square matrix M is in Jordan form if $M = \begin{pmatrix} J_{\lambda_1} & & \\ & J_{\lambda_2} & \\ & & \ddots \\ & & & J_{\lambda_k} \end{pmatrix}$ | $M = SJS^{-1}$ (Jordan)

Thm 2: M square then M is similar to some Jordan form matrix J

$$\left.\begin{aligned} &\left(\frac{d}{dt} - \lambda_1\right)^{m_1} y_1 = 0 \Rightarrow \text{Jordan cell} \\ &\left(\frac{d}{dt} - \lambda_2\right)^{m_2} y_2 = 0 \rightarrow \cdots \\ &\vdots \\ &\left(\frac{d}{dt} - \lambda_k\right)^{m_k} y_k = 0 \end{aligned}\right\} \text{Jordan System}$$

Chapter 2 :

$$\begin{pmatrix} \text{out} \\ \text{put} \end{pmatrix} = B \cdot A \begin{pmatrix} \text{input} \end{pmatrix}$$

2.1. premise :

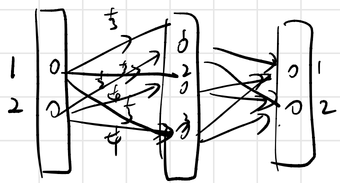
2.1.1 transpose

A : $n \times m$ matrix
$$\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & a_{n2} \cdots & a_{nm} \end{bmatrix}$$

A^T : $m \times n$ matrix
$$\begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1m} & \cdots & a_{nm} \end{bmatrix}$$

$(A^T)_{ij} = A_{ji}$

2.1.2. motivation examples



step1

out \ in	1	2
1	$\frac{1}{3}$	$\frac{1}{2}$
2	$\frac{1}{3}$	$\frac{1}{4}$
3	$\frac{1}{3}$	$\frac{1}{4}$

step2

out \ in	1	2	3
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
2	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{2}{3} & \frac{3}{4} \end{pmatrix}$$

2.2. determinant

• Def : A : $n \times n$ matrix
$$\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$$

$$\det(A) := \sum_{\sigma \in S_n} \varepsilon(\sigma) \cdot a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

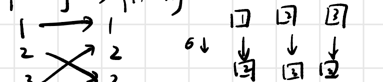
S_n : the group of permutation of $\{1, \cdots, n\}$ elements

$\uparrow$

1.1* A : def : a permutation of $\{1, \cdots n\}$ $\{S_n\}$ is a bijection

$$\sigma : \{1, \cdots n\} \to \{1, \cdots n\}, \quad i \mapsto \sigma(i)$$

ex 1.



$\sigma(1) = 2 \quad \sigma(2) = 3 \quad \sigma(3) = 2$

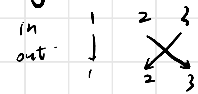
1.2* number of permutation of size n is $n!$

1.3* length of permutation :

$L(\sigma) \geq 0$ integer

1.3.1* geometrical def

$$\sigma \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$



$L(\sigma)$ = # of pairs (i, j) that (swap ordering) $i < j$ and $\sigma(i) > \sigma(j)$

= # crossings

= minimum number of elementary swapping to obtain σ

Ex: $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$L(\sigma) = 3$

Rmk: ① $L(\sigma) \le \binom{n}{2} = \frac{n\times(n-1)}{2}$

② There is exactly 1 element $\sigma \in S_n$ that $L(\sigma) = \frac{n(n-1)}{2}$

Def: $\varepsilon(\sigma) = (-1)^{L(\sigma)} = \begin{cases} +1 & L(\sigma) \text{ even} \\ -1 & L(\sigma) \text{ odd} \end{cases}$

$n=3$:

· $\sigma \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $L(\sigma)=0$, $\varepsilon(\sigma)=1$

$s_1 = s_{12}$ · $\sigma \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $L(\sigma)=1$, $\varepsilon(\sigma)=-1$

$s_2 = s_{23}$ · $\sigma \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $L(\sigma)=1$, $\varepsilon(\sigma)=-1$

} only 2 $\underbrace{L(\sigma)=1 \text{ permutation}}_{\text{simple transposition}}$.

· $\sigma \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $L(\sigma)=2$ $\varepsilon(\sigma)=1$

· $\sigma \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $L(\sigma)=2$ $\varepsilon(\sigma)=1$

· $\sigma \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ 3 $\varepsilon(\sigma)=-1$

	length	sign
$(1\,2\,3)$	0	+
$(2\,1\,3)$, $(1\,3\,2)$	1	−
$(2\,3\,1)$, $(3\,1\,2)$	2	+
$(3\,2\,1)$	3	−

$$\begin{tikzcd}[column sep=small] & (1\,2\,3) \arrow[dl, "s_{12}"'] \arrow[dr, "s_{23}"] & \\ (2\,1\,3) \arrow[d, "s_{23}"] & & (1\,3\,2) \arrow[d, "s_{12}"] \\ (2\,3\,1) \arrow[dr, "s_{12}"'] & & (3\,1\,2) \arrow[dl, "s_{23}"] \\ & (3\,2\,1) & \end{tikzcd}$$

By def 1: $\sum_{\sigma \in S_n} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$

For each summand each row & column appears once.

$\begin{matrix} \odot & \cdot & \cdot \\ \cdot & \odot & \cdot \\ \cdot & \cdot & \odot \end{matrix}$

$(1\,2\,3)$

def2: $\sum_{\sigma} \varepsilon(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}$

$\Downarrow$

$\begin{matrix} \cdot & \odot & \cdot \\ \odot & \cdot & \cdot \\ \cdot & \cdot & \odot \end{matrix}$

$(2\,1\,3)$

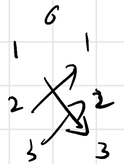
Proof def = alternate def 1

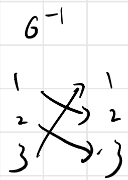
$n=3$

$$\det(A) = \sum_{\sigma} \varepsilon(\sigma)\, A_{1\sigma(1)} A_{2\sigma(2)} A_{3\sigma(3)}$$

$$= \sum_{\sigma} \varepsilon(\sigma)\, A_{\sigma^{-1}(1)1} A_{\sigma^{-1}(2)2} A_{\sigma^{-1}(3)3}.$$

ex:





$$\sum_{\sigma} \varepsilon(\sigma)\, a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}$$

$\because \sigma\sigma^{-1} = Id$ - even

$\therefore \varepsilon(\sigma) = \varepsilon(\sigma^{-1})$

rename σ^{-1} - σ

$$= \sum \varepsilon(\sigma^{-1})\, A_{\sigma(1)1} A_{\sigma(2)2} A_{\sigma(3)3}$$

$$= \sum \varepsilon(\sigma)\, A_{\sigma(1)1} A_{\sigma(2)2} A_{\sigma(3)3}$$

$\therefore$ permute by row = permute by column

$\Rightarrow$ Properties of det.

① $\det(A^t) = \det A$

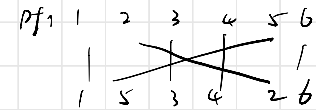
② Interchange any 2 columns would change sign

$$A = [\vec{a}_1, \vec{a}_2, \cdots \vec{a}_n]$$

$$\det([\vec{a}_1, \vec{a}_2, \cdots \vec{a}_n]) = (-1)\det[\vec{a}_1, \vec{a}_2 \cdots \overbrace{\vec{a}_j, \vec{a}_{i+1} \cdots \vec{a}_i}^{\text{swap } \vec{a}_i \text{ with } \vec{a}_j}, \vec{a}_{j+1} \cdots \vec{a}_n]$$

Lemma: if σ is a transposition (ij) then $\varepsilon(\sigma) = -1$

Pf:



$$\sigma(ij) \Rightarrow L(ij) = 2(j-i-1)+1$$
$$= \text{odd}.$$

③ $\det(\vec{a}_1, \vec{a}_2, \cdots \lambda\vec{a}_i, \vec{a}_{i+1}, \cdots \vec{a}_n)$

$= \lambda \cdot \det(\vec{a}_1, \vec{a}_2, \cdots \vec{a}_{i-1}, \vec{a}_i, \vec{a}_{i+1}, \cdots \vec{a}_n)$

for any $\lambda \in R$

④ $\det(\vec{a}_1, \vec{a}_2 + \vec{a}_2', \vec{a}_3)$

$= \det(\vec{a}_1, \vec{a}_2, \vec{a}_3) + \det(\vec{a}_1, \vec{a}_2', \vec{a}_3)$

③, ④ $\Leftrightarrow$ $\det(-,-,\dots,-)$ is a multilinear function from $\underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \mathbb{R}^n}_{n \text{ factor}} \to \mathbb{R}$

⑤ $\det I = 1$

Corrolary.

① $\det(\vec{a_1}, \cdots \vec{a_n}) = 0$ if one of the $\vec{a_i} = 0$

② $\det(\vec{a_1}, \cdots \vec{a_n}) = 0$, if there are i, j. s.t. $\vec{a_i} = \vec{a_j}$.

pf 1: properties

pf 2: permutation

$$\det(A) = \sum_{\sigma} (-1)^{\sigma} \underbrace{a_{\sigma(1)1} \cdots a_{\sigma(n)n}}_{S(\sigma)}$$

if $a_i = a_j$ then

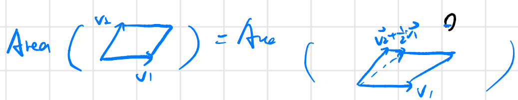
$$S(\sigma) = S(\sigma \cdot \overset{ij}{\cdot})$$

but $(-1)^{\sigma} = -(-1)^{\sigma(ij)}$

③. fix $i, j, \lambda \in \mathbb{R}$

$$\det(\vec{a_1}, \vec{a_2}, \cdots \vec{a_n})$$
$$= \det(\vec{a_1}, \vec{a_2}, \cdots \vec{a_i} + \lambda\vec{a_j}, \cdots \vec{a_n})$$

RHS $= \det(a_1, \cdots a_n) + \det(a_1, \cdots, \lambda\vec{a_j}, \cdots, \vec{a_j}, \cdots a_n)$.



④ $\det\begin{pmatrix} \lambda_1 & & \ddots \\ & \lambda_2 & \\ 0 & & \lambda_n \end{pmatrix} = \lambda_1\lambda_2\lambda_3 \cdots \lambda_n$

RHS $= \lambda_1 \cdots \lambda_n \begin{pmatrix} 1 & & \ddots \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \lambda_1 \cdots \lambda_n \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

pf 1: permutation by column

$$\begin{pmatrix} 1 & * & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & & & 1 \end{pmatrix}$$

pf 2: Gausian elimination.

block triangular matrix

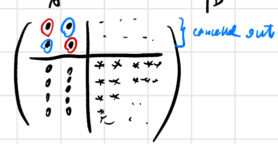
$$M = \begin{array}{c} n_1 \\ n_2 \end{array} \left[\begin{array}{c|c} \overset{n_1}{A} & \overset{n_2}{B} \\ \hline C & D \end{array}\right]$$

• if $C = 0$, the $\det M = \det A \cdot \det D$

(or $B = 0$ by symmetry)

$$\begin{array}{c|c} A & B \\ \hline 0 & D \end{array} = \begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}$$

$M =$



Multiplicativity

- Prop: if A, B are $n\times n$ matrices
  then $\det(A\cdot B) = (\det A)\cdot(\det B)$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "B"] \arrow[rr, bend right, "B\cdot A"'] & \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \end{tikzcd}$$

how much area change?

Pf: $\det(A)\cdot\det(B)$

$$= \det\underbrace{\left(\begin{array}{c|c} A & 0 \\ \hline -I & B \end{array}\right)}_{2n\times 2n}$$

$\star$ cancel out B

try ex1 $\begin{array}{c}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\\ c_1 \quad c_2\end{array}$ $\underset{=}{2\times c_1 + c_2} \Rightarrow \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$

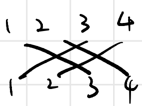
$\cdot b_{11}+$ $\quad \cdot b_{21}+$

ex2: $\left(\begin{array}{cc|cc} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ \hline -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \end{array}\right) \to \left(\begin{array}{cc|cc} a_{11} & a_{12} & b_{11}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline -1 & 0 & 0 & b_{12} \\ 0 & -1 & 0 & b_{22} \end{array}\right) \Leftrightarrow \left(\begin{array}{c|cc} & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline & & \end{array}\right)$

eventually $\Rightarrow$ $\left(\begin{array}{c|c} A & A\cdot B \\ \hline -1 & 0 \end{array}\right) = (-1)^{n^2} \det\left(\begin{array}{c|c} AB & A \\ \hline 0 & -I \end{array}\right)$

$\Leftrightarrow (-1)^n \det(AB)\cdot\det(-I_n) = (-1)^n\cdot(-1)^n \det(AB)\cdot\det(I)$
$= \det(AB)$

$\star$



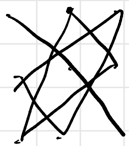
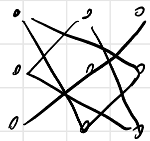
$\#$ crossing $\begin{pmatrix} 1\,2\cdots n, & n+1 \cdots 2n \\ n+1\cdots 2n, & 1\,2\cdots n \end{pmatrix} = n^2$.

Application:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det A = \sum_{\sigma \in S_3} \varepsilon(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)}$$

$$= a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$+$  $-$ 

"$S(n)$" set of all permutation group $[n] = \{1, 2, \dots n\}$

given $\sigma_1 : [n] \to [n]$

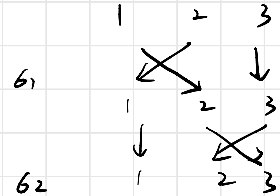
$\sigma_2 : [n] \to [n]$

I can compose them

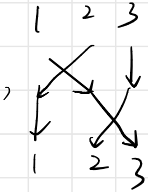
$$\sigma_2 \circ \sigma_1 : [n] \to [n]$$
$$i \mapsto \sigma_2(\sigma_1(i))$$

map of sets $f : A \to B$ $\quad g \circ f : A \to C$

$g : B \to C$ $\quad a \mapsto g(f(a))$



$(\sigma_2 \circ \sigma_1) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$



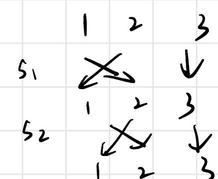
lemma: let $\sigma \in S(n)$, s_i be a simple transposition swapping $i, i+1$

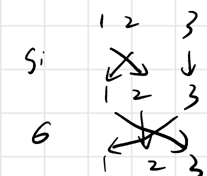
then $L(\sigma \cdot s_i) = \begin{cases} L(\sigma)+1 & \text{if } \sigma(i) < \sigma(i+1) \\ L(\sigma)-1 & \text{if } \sigma(i) < \sigma(i+1) \end{cases}$

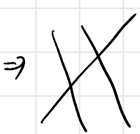
proof: consider any pair (j, k) with $j < k$

- if $(j,k) \neq (i, i+1)$ then $\sigma(j) < \sigma(k)$ $\Leftrightarrow \sigma s_i(j) < \sigma s_i(k)$
- if $(j,k) = (i, i+1)$ $\quad \sigma(j) < \sigma(j+1)$ $\Leftrightarrow \sigma s_i(j) > \sigma s_i(j+1)$

e.g.





$\Rightarrow$ 

$s_i \cdot s_i = \mathrm{Id}$.

Prop: if $\sigma \in S_n$ has length $L(\sigma)$, then

σ can be written as

$$\sigma = s_{i_{L(\sigma)}} \cdots s_{i_2} s_{i_1}$$

where $i_1, \cdots, i_L \in \{1, \cdots, n-1\}$

Pf: Proof by induction

Obviously, the statement is true for $L(\sigma) = 1, 0$.

Suppose the statement is true for all those σ, with $L(\sigma) \le k-1$

And suppose $L(\sigma) = k$

- Find a s_i, such that

$$L(\sigma \cdot s_i) = L(\sigma) - 1 = k-1$$

then by induction hypothesis

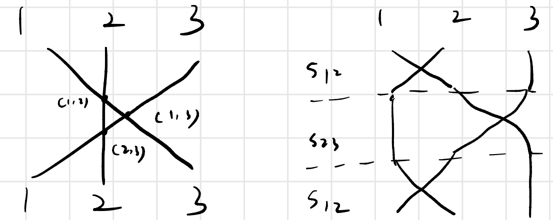
$$(\sigma s_i) = s_{i_{k-1}} \cdots s_{i_1}$$

$$\sigma = (\sigma s_i) \cdot s_i = s_{i_{k-1}} \cdots s_{i_1} \cdot s_i$$

- If no such s_i exists, then $\sigma(i) < \sigma(i+1)$ then $\sigma = id$ which we already cover

Pf by pic.

$$\sigma = s_{12} s_{23} s_{12}$$



1.1 linear forms / function

$a(\vec{x}) = a_1x_1 + a_2x_2 + \cdots a_nx_n = [a_1 \cdots a_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$a = [a_1 \cdots a_n]$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad K = \mathbb{R} / \mathbb{C}$

$a: k^n \to k.$

$a(\lambda x + \mu y) = \lambda a(x) + \mu a(y)$

1.2 linear map. An m-tuple of linear forms $a_1 \cdots a_m$ defines a linear map $A: k^n \to k^m$

$A = \begin{bmatrix} \cdots a_1 \cdots \\ \cdots a_2 \cdots \\ \vdots \\ \cdots a_m \cdots \end{bmatrix}$ $\quad A(\lambda x + \mu y) = \lambda A(x) + \mu A(y)$

property: $\begin{cases} \text{row: linear forms} \\ \text{the } j\text{th column of } A \text{ is just } Ae_j. \end{cases}$

example: $z = By \quad y = Ax$

$A: K^n \to k^m \quad B: k^m \to k^q$

$z = (BA)x$

$BA: K^n \to k^q$

2. Determinant

Def 2: $M_n(k) \to k.$ a function from matrix to numbers of same field of M

s.t.
1) $\det(I_n) = 1$
2) If $B = A$ except $row_i(B) = c \cdot row_i(A)$ then $\det(B) = c \cdot \det(A)$
3) If A, B, C are all equal except $row_i(A) = row_i(B) + row_i(C)$ then $\det(A) = \det(B) + \det(C)$

Linear in each row

4) If B arises from A by swapping two rows then $\det(B) = -\det(A)$

Remark: $\det(M)$ can be think of as a linear function of rows

$= \det(row_1(M), row_2(M_2), \cdots row_r(M))$

$\det(\lambda row_1(M) + \mu row_1(M'), \cdots row_r(M))$

$= \lambda \det(M) + \mu \det(row_1(M'), row_2(M), \cdots row_r(M))$

multilinear

Theorems:

properties: a) the determinant of any matrix w/ a row of all zeros is 0

Def 2) $\Leftarrow$ b) ~ two identical row is 0

c) If one row is the multiple of another row, then ~ 0.

d) If a multiple of one row is added to another, then the resulting matrix has the same determinant.

d) $M' = \begin{bmatrix} \text{row}_1 \\ \vdots \\ \text{row}_i + c \cdot \text{row}_j \\ \vdots \\ \text{row}_r \end{bmatrix} = \underbrace{\begin{bmatrix} \text{row}_1 \\ \vdots \\ \text{row}_i \\ \vdots \\ \text{row}_r \end{bmatrix}}_{M} + \underbrace{\begin{bmatrix} \text{row}_1 \\ \vdots \\ c\,\text{row}_j \\ \vdots \\ \text{row}_r \end{bmatrix}}_{A}$

$A = 0$ by c)

above is just existence, but lacks uniqueness

Theorem: Uniqueness of the determinant

Proof: Let A be a matrix with at least one non-zero entry in each row and each column

By multilinearity we know $\det(A) = \det(A_1) + \cdots + \det(A_n)$

where A_j is the matrix that is identical to A

except in the 1st row it's all 0s except for the jth element

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ 0 & - & - & - \\ & & - & - \\ & & - & - \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}$

$\Downarrow$

$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

$\because$ $\det(I)$ is unique,

& each step is invertible.

$\therefore$ uniquely 0 or 1

Cofactor

- motivation

$A \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \det(A) = ad - bc$.

if $\det(A) \neq 0$, $\quad A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

- Def.

A: $n \times n$ matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \\ A_{n1} & \cdots & & A_{nn} \end{bmatrix}$$

Given $1 \le \underset{\text{row}}{i}, \underset{\text{column}}{j} \le n$, we form a new (sub) matrix, by deleting the ith row and jth column, we get $[A]_{\hat{i}\hat{j}}$

let $\underbrace{c_{ij}}_{\text{cofactor}} = (-1)^{i+j} \det((n-1) \times (n-1) \text{ matrix})$

Thm: ① $\det(A) = A_{11} \cdot C_{11} + A_{12} C_{12} + \cdots A_{1n} C_{1n}$ (1st row)

$= A_{11} \cdot C_{11} + A_{21} C_{21} + \cdots A_{n1} C_{n1}$ (1st col)

$$C_{ij} \Rightarrow C = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}$$

in general, for $1 \le i_1, i_2$, we have

$$\sum_{i=1}^{n} A_{i_1 j} C_{i_2 j} = \begin{cases} \det(A) & i_1 = i_2 \\ 0 & i_1 \neq i_2 \end{cases}$$

$$= \det(A) \cdot \delta_{i_1 i_2} \left(\begin{array}{cc} & i_2 \\ i_1 & \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \ddots \end{bmatrix} \end{array} \right)$$

similarly $\sum_{j=1}^{n} A_{j i_1} C_{j i_2} = (\delta_{i_1 i_2}) \cdot \det A$

pf: Thm ①.

$$\det(A) = \sum_{\sigma} \varepsilon(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$$

pic

$$\begin{matrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{matrix}$$

$$\begin{bmatrix} \text{row}_1 \\ \text{row}_2 \\ \vdots \\ \text{row } i \\ \vdots \\ \text{row } n \end{bmatrix} \quad \begin{bmatrix} \text{row} \end{bmatrix}$$

$$= \underbrace{A_{11} \cdot \det(\overset{\text{matrix } A}{\text{delete 1-th row 1st col}})}_{\text{all the term with } \sigma(1)=1}$$

$$- \underbrace{A_{12} \det(\quad C_{12} \quad)}_{\text{all the term with } \sigma(1)=2}$$

why negative? $+ \quad A_{13} C_{13} - \cdots + A_{1n} C_{1n}$

① swap A_{11} A_{12}.

$\therefore$ $\det A_{11} = +$

$\det A_{12} = -$

②
$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ & \times & | & | & | \\ 1 & 2 & 3 & 4 & 5 \end{matrix}$$

If $i_1 = i_2 = i$ $\Rightarrow$ $A_{i1} C_{i1} + A_{i2} C_{i2} + \cdots + A_{in} C_{in} = \det A$

Pf: build a new matrix $\tilde{A}$ that moves the i-th row to the top
swap $i-1$ times

$\det(\tilde{A}) = (-1)^{i-1} \det(A) \quad (\tilde{A})_{1j} = A_{ij}$

$\tilde{C}_{1j} = (-1)^{1+j} \det(\tilde{A} \text{ deleting } \begin{matrix}\text{1st row} \\ j\text{th column}\end{matrix})$

$= (-1)^{1+j} \det(A \text{ deleting } \begin{matrix}i\text{th row} \\ j\text{th col}\end{matrix})$

$$= \underbrace{(-1)^{1+j} (-1)^{i+j}}_{(-1)^{i+1}} \underbrace{(-1)^{i+j} \det(\cdots)}_{C_{ij}}$$

$$\det \tilde{A} = \sum_j \tilde{A}_{1j} \tilde{C}_{1j}$$

$$(-1)^{i+1} \det A = \sum_j A_{ij} C_{ij} (-1)^{i+1}$$

if $i_1 \neq i_2$, say $i_1 = 1$, $i_2 = 2$, we want to show that

$$\sum_{j=1}^{n} A_{1j} \cdot C_{2j} = 0$$

$$A = \begin{bmatrix} \boxed{A_{11} \ A_{12} \cdots A_{1n}} \\ \\ \\ \end{bmatrix} \qquad C = \begin{bmatrix} C_{11} \ C \cdots \\ \boxed{C_{21} \ C_{22} \cdots \ C_{2n}} \\ \\ \end{bmatrix}$$

$\tilde{A}$ copies the ith row of A

for $i = 2$, $\tilde{A}_{ij} = A_{ij}$

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{11} & A_{12} & \cdots & A_{1n} \\ A_{31} & \cdots & & \\ & & & \end{bmatrix}$$

$$0 = \det(\tilde{A}) = \tilde{A}_{21}\tilde{C}_{21} + \tilde{A}_{22}\tilde{C}_{22} + \cdots \tilde{A}_{2n}\tilde{C}_{2n}$$
$$= A_{11}C_{21} + A_{12}C_{22} + \cdots A_{11}C_{2n}$$

Corollary: if A is an $n \times n$ matrix, with $\det A \neq 0$

then $A^{-1} = \frac{1}{\det A} C^t$ $\qquad C^t = \text{adjoint}(A)$

pf:

$$\left[\left(\frac{1}{\det A} C^t\right) \cdot A\right]_{ij}$$

$$= \frac{1}{\det A} \cdot \sum_{k=1}^{n} (C^t)_{ik} A_{kj} = \frac{1}{\det A} \sum_{k=1}^{n} C_{ki} A_{kj} = \frac{1}{\det A} \cdot \det A \cdot \varepsilon_{ij} = \delta_{ij}$$

$$= \left[A \cdot \left[\frac{1}{\det} C^t\right]\right]_{ij} \overset{!}{=} \delta_{ij}$$

Cramer's Rule:

- let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be a column of unknown variables

  let $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ be a given vector.

  let $A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$ be an invertible $n \times n$ matrix

- consider equation $A\vec{x} = \vec{a}$

  then $\vec{x} = A^{-1} \cdot \vec{a}$

thus.

$$x_1 = \frac{\det \begin{pmatrix} a_1 & \overset{A_{\cdot 2}}{\begin{pmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{n2} \end{pmatrix}} & \cdots \\ \vdots & & \cdots \\ a_n & & \cdots \end{pmatrix}}{\det A} = \frac{\det(\vec{a}, A_{\cdot 2}, A_{\cdot 3} \cdots A_{\cdot n})}{\det A}$$

$\vdots$

$$x_i = \frac{\det(A_{\cdot 1}, \cdots \overset{i\text{th slot}}{\vec{a}} \cdots A_{\cdot n})}{\det A}$$

Ex:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$x_1 = \frac{\det \begin{pmatrix} a_1 & A_{12} \\ a_2 & A_{22} \end{pmatrix}}{\det(A)} \qquad x_2 = \frac{\det \begin{pmatrix} A_{11} & a_1 \\ A_{21} & a_2 \end{pmatrix}}{\det(A)}$$

Prove by finding a bigger matrix

Pf:

$$\begin{aligned} x_i &= (A^{-1} \cdot \vec{a})_i \\ &= \sum_{k=1}^{n} (A^{-1})_{ik} a_k \\ &= \sum_{k=1}^{n} \left(\frac{1}{\det A} C^t\right)_{ik} a_k \\ &= \frac{1}{\det A} \left[\sum_{k=1}^{n} C_{ki} \cdot a_k \right] = \frac{1}{\det A} \cdot \det \begin{pmatrix} A, \text{ replacing the} \\ i\text{th column by } \vec{a} \end{pmatrix} \end{aligned}$$

3 cool formulas

1. Block.

$$A = \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c|c} n_1 & n_2 \\ \hline A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \qquad B = \begin{array}{c|c} n_1 & n_2 \\ \hline B_{11} & B_{12} \\ B_{21} & B_{22} \end{array}$$

$$AB = \left(\begin{array}{c|c} A_{11}B_{11} + A_{12} \cdot B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline \cdots & \cdots \end{array}\right)$$

Rmk: One can treat the blocks as if they are numbers.

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

if $\det D \neq 0$

$$\det(M) = \det(D) \cdot \det(A - \underbrace{B \cdot D^{-1} C}_{\text{size } n_1 \times n_1})$$

$$\left[M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \det M = ad - bc = d\left(a - \frac{bc}{d}\right) \right]$$

$d \neq 0$

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \det\left(\begin{array}{c|c} I & 0 \\ \hline -D^{-1}C & I \end{array}\right)$$

$$= \det\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix}\right]$$

$$= \det\begin{bmatrix} A - BD^{-1}C & B \\ C - DD^{-1}C & D \end{bmatrix}$$

$$= \det\begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix}$$

$$= \det(A - BD^{-1}C)\det(D)$$

3. Vector spaces.

1. Field.

Recall. algebra — Field → a set K with $+, \times, 0, 1$, s.t.:
module — vector space

① $(K, +, 0)$ every element in K
$\forall x, \exists x' \in K, x + x' = 0$

② $(K^* = K \setminus \{0\}, \cdot, 1)$
$\forall y, \exists y' \in K^*, y \cdot y' = 1$

2. Examples: $\mathbb{R}, \mathbb{C}, \mathbb{Q}$,

module.

p. prime number. $\mathbb{F}_p = \mathbb{Z}/p\cdot\mathbb{Z} = \{0, 1, 2 \cdots p-1\}$

e.g. $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$.

$2+3=5 \to 0$

$1+4=5 \to 0$.

$1 \cdot 1 = 1$
$2 \cdot 3 = 1$
$3 \cdot 2 = 1$
$4 \cdot 4 = 1$

3. Field extension

Q: does $\mathbb{Z}/6\mathbb{Z}$ form a field under the natural $(+, \cdot)$

A: No. $6+1$ is a prime number

$2 \cdot x = 1 \pmod 6$

$2 \cdot x = 1 \pmod 2$ contradict.

4. p. prime number

$n \geq 1$ integer

prop: $\exists!\ \mathbb{F}_{p^n}$: finite field with p^n elements.

pf:

Let $f(x) \in \mathbb{F}_p[x]$ with variable / coefficient in $\mathbb{F}_p$.
polynomial.

e.g. if $p=5$ $f(x) = 3x^2 + 4x - 2 \in \mathbb{F}_5[x]$

Assume. $f(x)$ is irreducible in $\mathbb{F}_p[x]$,
i.e. $f(x) \neq g(x) \cdot h(x)$ w/ g and h in smaller degrees

prop: then $\cancel{\mathbb{F}_p[x]}$ $f(x) \cdot \mathbb{F}_p[x]$ is a field of size p^n.

Ex:

pf: $p=5$. $f(x) = x^2+1$
$= (x-2)(x-3)$

$0^2 = 0$
$1^2 = 1$
$2^2 = 4$
$3^2 = 4$
$4^2 = 1$

$f(x) = x^2 + 2$

on top of $5 \Rightarrow$

$F_{5^2} = F_5[x]/(x^2+2) \rightarrow$ declare $x^2+2=0$

typical element: $x,\ 3,\ x^5+1$

$x^2+3 = (x^2+2)+1 = 1$

$x^2 = -2$

$x^3 = x^2 \cdot x = -2x$

as. vector space

$F_5[x]/(x^2+2) = \{ax+b \mid a, b \in F_5\} \simeq (F_5)^2 = 25$

$\mathbb{C} \simeq R^2$

e.g. $x \cdot 2x = 1$

5.

$\mathbb{Q}[\sqrt{-1}] := \mathbb{Q}[x]/x^2+1 \simeq \mathbb{Q}^2$

$= \{ax+b \mid a, b \in \mathbb{Q}\}$

$(ax+b)(cx+d) = -(ac+bd) + (bc+ad) \cdot x \cdot$

$x \cdot \frac{x}{2} = 1$

2. Let K be a field

A vector space V over K is a set s.t.:

① $(V, +, 0)$ is an abelian group

$- \ V \times V \to V$

② $K \circlearrowright V$ (scalar product)

$- \ K \times V \to V$

③ Given two vector space V, W

$\mathrm{LinMap}(V, W)$ is the set of <u>linear maps</u>

$\Leftrightarrow \mathrm{Hom}(V, W)$ also forms a vector space over K

④ Direct Sum

$V \oplus W = \{(v, w) \mid v \in V,\ w \in W\} = (V \times W)$

$(v_1, w_1),\ (v_2, w_2) \in V \oplus W$

$(v_1, w_1) + (v_2, w_2) = (v_1+v_2,\ w_1+w_2)$

$R \oplus R = R^2$

⑤ $c \in \mathbb{K}$

$c(v, w) = (cv, cw)$

⑥ Quotient Space

Let V be a vector space, $W \subset V$ a subspace

then V/W is the quotient space consisting of

equivalence class in V where we say $v_1, v_2 \in V$ are equivalent

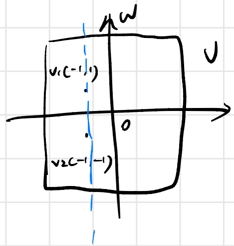
if $v_1 - v_2 \in W$ one can be related to another by adding an element in W.

Ex: $V = \mathbb{R}^2$

$W = \{(0,y) \mid y \in \mathbb{R}\}$

$\because [x] = \{x+n, \ n \in N\}$

$[x] = x + N$

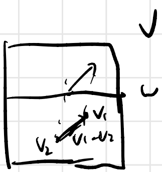


$v_1 = (-1,1)$

$v_2 = (-1,-1)$

$v_1 - v_2 = (0,2) \in W$

we say $v_1 \sim v_2$ ↪ equivalent.



★ Line: $v_1 + W = \{v_1 + w \mid w \in W\}$ is the equivalence class that v_1 belongs to

V/W = the set of vertical lines

Let $v_1 + W$, $v_2 + W \in V/W$ (e.c.)

then $(v_1 + W) + (v_2 + W)$ $[x]+[y] = [x+y]$

$= (v_1 + v_2) + W$ is well defined

because

if $\tilde{v}_1 + W = v_1 + W$ i.e. $\tilde{v}_1 = v_1 + w$

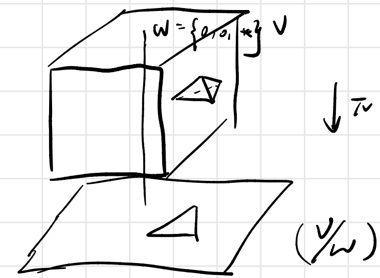
$\tilde{v}_2 + W = v_2 + W$ $\tilde{v}_2 = v_2 + w$

then $\tilde{v}_1 + \tilde{v}_2 \in v_1 + v_2 + W$

if $v_1 - \tilde{v}_1 \in W$

$v_2 - \tilde{v}_2 \in W$

then $(v_1 + v_2) - (\tilde{v}_1 + \tilde{v}_2) \in W$



Quotient map: $\pi: V \longrightarrow V/W$

$v \longmapsto v + W$

if. V is a $\mathbb{R}$ vector space and has inner product

$\langle -,- \rangle : V \times V \to \mathbb{R}$.

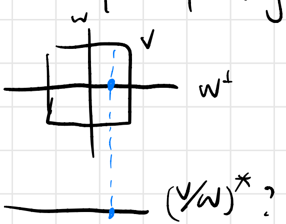
then given $W \subset V$, we can form $W^\perp \subset V$

recall that: given $\langle -,- \rangle$

$v_1 \perp v_2 \iff \langle v_1, v_2 \rangle = 0$

$v_1 \perp W \iff \forall w \in W,\ v \perp w$
$\langle v, w \rangle = 0$

$W^\perp = \{ v \in V \mid v \perp W \} \subset V$



$\pi : V \to V/W$

π restricted to $W^\perp$ gives a bijection

$\pi|_{W^\perp} : W^\perp \to V/W$

coordinate system k^n = set of n tuples of numbers in k

= set of all sequences of n number

k^S : sets of all $f: S \longrightarrow K$

$S = \{1, 2, \cdots, n\}$ $\quad f(s): \begin{matrix} 1 \to \\ 2 \to \\ \vdots \\ n \to \end{matrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in K$.

$v \in K^S \quad v(s) = k \in K$

$v : \begin{cases} v(1) = \text{—} \\ v(2) = \text{—} \\ \vdots \\ v(n) = \text{—} \end{cases}$ set of all of v is in K^S -

Axiom: • function = map on a set S, valued in $\mathbb{K}$ is an assignment to each element in S, some element in k

The set of all functions from S to K.
is $\text{map}(S, \mathbb{K}) = K^S$

Ex: if $S = \{*\}$
$K^S = K$

if $S = \{a, b\}$
$K^S = K^2$

if $s = \emptyset$
$K^\emptyset = K^0 = \{pt\}$

$f, g : \{1, 2\} \to \mathbb{R}$, $f, g \in Map(\{1,2\}, K)$

- $(f+g)(x) = f(x) + g(x)$, $\forall x \in \{1,2\}$
- $\forall c \in K$, $(c \cdot f)(x) = cf(x)$.

subspace.

$S = \{1, 2, \cdots n\} = [n]$ $K^S = K^n$

$S = [n] \times [m]$

$= \{(i,j) \mid 1 \le i \le n, 1 \le j \le m, \text{int}\}$ $K^S = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \middle| a_{ij} \in K \right\}$ vector space

- let V be a vector space over K.
  A subspace $W \subset V$ is a subset that is closed under $+$ and K.

Ex: $K = \mathbb{R}$ $V = \mathbb{R}^{\mathbb{R}} = Map(\mathbb{R}, \mathbb{R})$ all functions on $\mathbb{R}$

$W = \mathbb{R}[x] = \left\{ a_n x^n + \cdots + a_0 \mid a_i \in \mathbb{R}, n \in \mathbb{Z}_{\ge 0} \right\}$ polynomials.

$\ker A$ trivial $\Leftrightarrow$ $v = \vec{0}$ $\Leftrightarrow$ linear independence.

Category Theory

- a set of objects
- morphisms between objects
- composition of morphisms

Ex 1: **Set**: the cat of sets.

- objects: any set.
- morphisms: Given S_1, S_2 two sets: $(S_1, S_2) = Map(S_1, S_2)$
- compositions.

$$\underset{f_{12}}{Mor(S_1, S_2)} \times \underset{f_{23}}{Mor(S_2, S_3)} \to Mor(S_1, S_3)$$

Vect: the category of vector spaces

- an object is a vector space over K
- a morphism between V_1, V_2 is a linear map $Mor(V_1, V_2) = LinMap(V_1, V_2)$

note: the set $LinMap(V_1, V_2)$ also forms a vector space

- let $f: V \to W$ be a linear map

nullspace
- $kernel(f) = \{ v \in V \mid f(v) = 0 \} \subset V$
- $im(f) = \{ w \in W \mid \exists v \in V, w = f(v) \} \subset W$
- $coker(f) = \dfrac{W}{im(f)}$

⑦ Dual space:

V v.s./$\mathbb{K}$

$V^* = \mathrm{Hom}(V, \mathbb{K})$

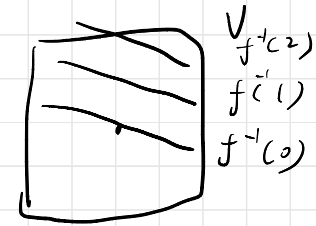
$=$ the set of linear functions on V.

Ex: $V = \mathbb{R}^2$

$V^* = \{ a x_1 + b x_2 \mid a, b \in \mathbb{R} \}$

x_1 x_2

①



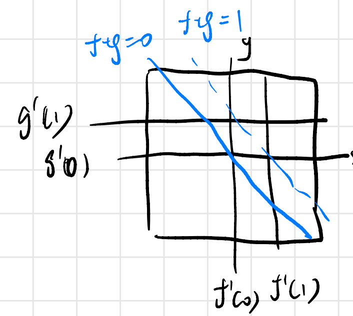
$f: V \to \mathbb{R}$.

Visualize f using level set

Knowing $f^{-1}(0)$ is enough.

②

V



$f: V \to \mathbb{R}$

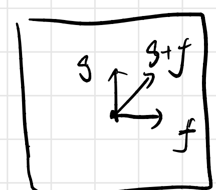
$g: V \to \mathbb{R}$

$f(x,y) = x$

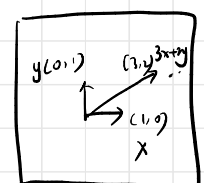
$g(x,y) = y$

$(f+g)(x,y) = x+y$

V^*



$f_2(x,y) = 3x+2y$



Chap 3.

1. Basis. span; linear independence

Let V be a vector space $/ \mathbb{K}$.

Def: a collection (possibly infinite) of vectors $v_1, v_2 \cdots$ is called a basis if any $v \in V$ $\boxed{\text{can}}$ be written $\boxed{\text{uniquely}}$ as a $\boxed{\text{finite}}$ linear combination of $\{v_i\}$

Ex: $\mathbb{K}[x]$ vector space of 1-variable polynomials

$\{1, x, x^2 \cdots\}$ forms a basis

Property: B is a basis $\Longleftrightarrow$ B is linearly independent. & B spans V

2. Span:

Let $v_1, v_2 \cdots v_n$ be vectors in V.

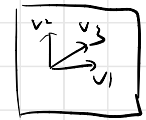
then we have a map

$$f: K^n \to V$$

$$(a_1, \cdots, a_n) \longmapsto a_1v_1 + a_2v_2 + \cdots a_nv_n$$

$$\text{span}\{v_1, \cdots, v_n\} = \text{image of } f = \{a_1v_1 + \cdots + a_nv_n \mid a_i \in K\}$$

we say $\{v_1 \cdots v_n\}$ span V if $\text{span}\{v_1 \cdots v_n\} = V$.

Ex: 

$V = \mathbb{R}^2$

$\text{span}(v_1, v_2, v_3) = V$

3. Linear independence.

- $\{v_1, \cdots v_n\}$ is linearly independent if the map $f: K^n \to V$ is injective.

(choosing different coefficients give different span)

$0 \longrightarrow 0$.

i.e. if $(a_1 \cdots a_n) \neq (0, \cdots 0)$

then $a_1v_1 + \cdots a_nv_n \neq 0$.

反例 v_2 $(2, 0, 0)$ v_1 $(1, 0, 0)$

$$2v_1 + (-1)v_2 = 0$$

例2: if $v_3 = a_1v_1 + a_2v_2$ for some a_1, a_2

$a_1v_1 + a_2v_2 + (-1)v_3 = 0$

Property of Basis

- $\dim_{\mathbb{K}} V =$ # of elements in a basis

- any collection of linearly indep vectors in finite-dim V
  $\{v_1, \dots, v_m\}$
  can be completed to a basis.

Exercise 1:

let $v_1 = (1, 1, 0)$
$v_2 = (1, 0, 1)$ in $\mathbb{R}^3$
$v_3 = (0, 1, 1)$

linearly independent. $\iff$ $\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \neq 0$

Ex 2.

$v_4 = (1, 1, 1)$

$\because \{v_1 v_2 v_3\}$ forms a basis

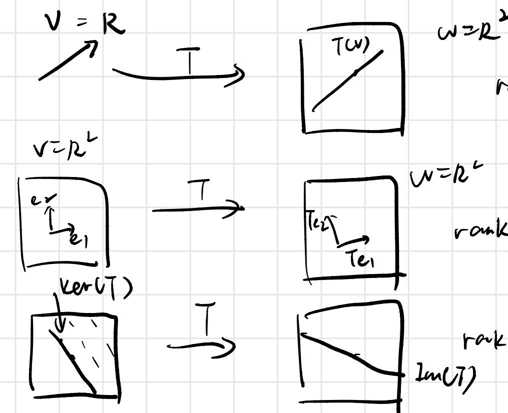
$\therefore \{v_1, v_2, v_3, v_4\}$ are linearly dependent

$T: V \to W$ $\quad V, W$ fin-dim

$\operatorname{rank}(T) := \dim(\operatorname{im} T)$

$\Rightarrow \operatorname{rank}(T) \leq \dim W$

$\operatorname{rank}(T) \leq \dim V$



$\operatorname{rank}(T) = 1$

$\operatorname{rank}(T) = 2.$

$\operatorname{rank}(T) = 1$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, two heads, "\pi"'] & W \\ V/\ker(T) \arrow[r, "\simeq"] & \operatorname{im}(T) \arrow[u, hook] \end{tikzcd}$$

surjection

injection

$$\operatorname{rank}(T) = \dim(\operatorname{im} T) = \dim(V/\ker T)$$

$$\dim(\operatorname{im} T) \leq \dim W, \qquad \dim(V/\ker T) \leq \dim V$$

$T^{-1}(w)$ is invariant under translation by $\ker(T)$

$\Leftrightarrow \quad \forall v \in T^{-1}(w), \ \tilde{v} \in \ker(T)$

1. Finite dim

1.1

Def: V is finite dimensional if $\exists$ a finite subset $\{v_1, \dots, v_m\}$ that spans V.

Lemma: if V is fin. dim, then $\exists$ a basis B of finite vectors

1.2 redundancy:

A vector $v \in S \subset V$ is redundant, if

$$\text{span} S = \text{span} S \setminus \{v\}$$

or $$v \in \text{span}(S \setminus \{v\})$$

1.3.

Let $B = \{v_1 \cdots v_n\}$ be a basis of V

let $\Phi : \mathbb{K}^n \to V$ be given by

$$(x_1, \dots x_n) \mapsto x_1 v_1 + \cdots x_n v_n$$

Then Φ is an isomorphism of v.s.

Lemma: any $n+1$ vectors in K^n are linearly dependent

Pf: For $n=1$, the statement says any 2 vectors in K are lin. dep. (numbers) in K

- If one of the vector is 0

$\Rightarrow$ $v_1 = 0$

$1 \cdot \vec{v}_1 = 0$

$\Rightarrow$ $\{v_1, v_2\}$ is lin. dep.

- if $v_1 v_2 \neq 0$ in K

$\Rightarrow$ $\left(\frac{v_1}{v_2}\right) \cdot v_2 - v_1 = 0$

Suppose the statement is true for $n = 1, 2, \cdots n_0 - 1$

we want to show $n = n_0$

Let $\{v_1, \cdots v_{n+1}\}$ be $n+1$ vectors in K^n

write them as column vectors.

$$\begin{array}{cc} v_1 \cdots & v_{n+1} \\ \left(\begin{matrix} v_{11} & \cdots & v_{n+1,1} \\ \vdots & & \vdots \\ v_{1n} & \cdots & v_{n+1,n} \end{matrix}\right) \end{array}$$

Lemma : ∘ if all the last coordinates of v_i are zero,
then $v_1, \cdots v_{n+1} \in K^{n-1}$

by induction we know $\{v_1 \cdots v_n\}$ are lin. dep.

∘ if not, by rearranging the order of columns
we assume $v_{n+1,n} \neq 0$.

then $\tilde{v}_1 = v_1 - \dfrac{v_{1n}}{v_{n+1,n}} \cdot v_{n+1}$

$\vdots$ $\qquad c_n$

$\tilde{v}_n = v_n - \dfrac{v_{nn}}{v_{n+1,n}} v_{n+1}$

then $\tilde{v}_{i,n} = v_{i,n} - \dfrac{v_{in}}{v_{n+1,n}} v_{n+1,n} = 0$.

$\Rightarrow \{\tilde{v}_1, \cdots \tilde{v}_n\}$ are in K^{n-1}

by induction, $\Downarrow$ lin. dep.

$\Rightarrow \exists (a_1, \cdots a_n) \in K^n$, non-zero

s.t. $a_1\tilde{v}_1 \cdots + a_n\tilde{v}_n = 0$

$\Rightarrow a_1(v_1 - c_1 v_{n+1}) + \cdots + a_n(v_n - c_n v_{n+1}) = 0$.

$\Leftrightarrow a_1 v_1 + \cdots + a_n v_n + (-a_1c_1 - \cdots - a_n c_n)\cdot v_{n+1} = 0$.

at least one of a_i is non-zero

Ex.

$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$
$$\begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & 3 \end{pmatrix} \quad \text{lin. dep. in } \mathbb{R}^2$$

$\tilde{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$

$\tilde{v}_2 = \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \frac{5}{3}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ 0 \end{pmatrix}$

$\Rightarrow 7\cdot\tilde{v}_1 - \tilde{v}_2 = 0$

Corollary : (1) for $\forall m > n$
any m vectors in K^n are Lin. dep.

(2) if $\exists$ isomorphism $T: K^n \to K^m$
then $n = m$

Pf: if not. suppose $n > m$, then $\{e_1 \cdots e_n\}$
basis in K^n is sent to $\{Te_1, \cdots Te_n\}$
in K^m

Since $n > m$ by lemma (1).

$\exists$ a non-zero linear combination
$a_1Te_1 + \cdots a_nTe_n = 0$ in K^m where $a_1 \neq 0$

$\because T$ invertible

$\therefore$ apply T^{-1} on both sides

$a_1e_1 + \cdots + a_ne_n = 0$

Impossible. since $\{e_i\}$ lin. indep.

Proof of Rank Theorem

(3) for any finite-dim v.s. V.
∃ unique n s.t. there is an isom
$$T: k^n \to V.$$

Pf: ∘ existence of $T: k^n \to V$ is guaranteed
by existence of basis.

∘ uniqueness:
if $m \neq n$ by corollary (2)

n here $=: \dim_k V$

(4). if 2 vector spaces V, W have the same dim.
then ∃ isom $T: V \to W$

① coefficients of L.C.
② send coordinate to vectors

$$T_1: \underline{k^n \xrightarrow{\sim} V}$$
$$\downarrow \simeq \text{ (identity map.)}$$
$$T_2 \quad k^n \xrightarrow{\sim} W$$

$=$ equal.
$\simeq$ isom
$\xrightarrow{\sim}$ isomorphic

③ basis in $k^n \to$ basis in V

concretely, pick basis $\{v_1, \cdots v_n\}$ of V
$\{w_1, \cdots, w_n\}$ of W

define T s.t. $T(v_i) = w_i$

Remark: T depends on choice of v_i, w_i
∴ isom $\neq$ unique.

Rank Theorem.

Let $T: V \to W$ be a linear map (over K) from V (finite-dim) to W

Let $r = \mathrm{rank}(T) := \dim_K(\mathrm{im}(T))$

Then there exists a basis $\{v_1, \cdots, v_n\}$ of V

$\{w_1, \cdots, w_m\}$ of W

s.t.

$$T(v_1) = w_1$$
$$\vdots$$
$$T(v_r) = w_r$$
$$T(v_{r+1}) = 0$$
$$\vdots$$
$$T(v_n) = 0$$

$\Longleftrightarrow$ in coordinates on V, W generated by the basis

$$K^n \to K^m$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto m\left\{ \underbrace{\overset{r}{\begin{pmatrix} \ddots & 0 \\ 0 & 0 \end{pmatrix}}}_{n} \right. \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\underset{K^n}{E_r\begin{pmatrix} \vdots \end{pmatrix}} = \underset{K^m}{\begin{pmatrix} \vdots \end{pmatrix}}$$

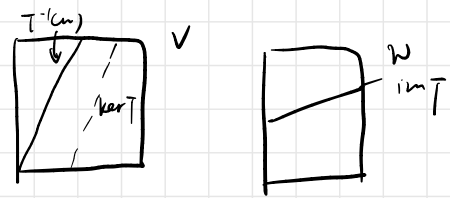
$$\begin{tikzcd} K^n \arrow[r, "E_r"] \arrow[d, "\simeq"] & K^m \arrow[d, "\simeq"] \\ V \arrow[r, "T"] & W \end{tikzcd}$$

Pf:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\text{surjective}"'] & W \\ (V/\ker T) \arrow[r, "\sim"] & \mathrm{im}\, T \arrow[u, "\text{injective}"'] \end{tikzcd}$$

For any $w \in \mathrm{im}\, T$,

$T^{-1}(w)$ is an equivalence class in $V/\ker T$



find a basis $\{w_1, \cdots w_r\}$ for $\mathrm{im}\, T$

complete the basis $\{w_1, \cdots w_r\}$ of $\mathrm{im}\, T$

$\{w_1, \cdots w_m\}$ a basis of W

pick $v_1 \in T^{-1}(w_1) \cdots v_r \in T^{-1}(w_r)$

let $\{v_{r+1}, \cdots v_n\}$ be the basis for $\ker T$

Claim: $\{v_1,\cdots v_r, v_{r+1},\cdots v_n\}$ forms a basis of V.

$$(\ker T) \hookrightarrow V \twoheadrightarrow V/\ker T$$

$\{v_{r+1},\cdots v_n\}$ $\quad$ $\{v_1\cdots v_r, v_{r+1}\cdots v_n\}$ $\quad$ $\{[v_1],\cdots [v_r]\}$

For any $v\in V$, consider its image $[v]\in V/\ker T$

then $[v]=a_1[v_1]+\cdots+a_r[v_r]$ in $V/\ker T$ for some a_i

thus $v-(a_1v_1+\cdots+a_rv_r)\in\ker T$

since $\{v_{r+1},\cdots v_n\}$ is a basis of $\ker T$,

we know $\exists!\ a_{r+1},\cdots a_n$, s.t.

$$v-(a_1v_1+\cdots+a_rv_r) = a_{r+1}v_{r+1}+\cdots+a_nv_n$$

Thus, $v=a_1v_1+\cdots+a_nv_n$

Remark: rank is the unique number that characterizes a linear map (isom)

Corollary: If V, V' of dim n, W, W' of dim m and $T: V\to W$, $T': V'\to W'$ of rank r

then $\exists$ isom $V\simeq V'$, $W\simeq W'$

s.t.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "A"', "\simeq"] & W \arrow[d, "B"', "\simeq"] \\ V' \arrow[r, "T'"] & W' \end{tikzcd}$$

s.t. $T=(B)^{-1}\circ T'\circ A \to$:

Pf:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\simeq"] & W \arrow[d, "\simeq"] \\ K^n \arrow[r, "E_r"] & K^m \\ V' \arrow[r, "T'"] \arrow[u, "\simeq"] & W' \arrow[u, "\simeq"] \end{tikzcd}$$

Rank Thm: $\forall$ linear map $A: \underset{\dim n}{V} \to \underset{\dim m}{W}$ of rank r

is given by E_r matrix with suitable bases

Concretely, $\forall$ $m\times n$ matrix A of rank r.
$\exists$ invertible matrix D and C of size $m\times m$, $n\times n$
s.t. $D^{-1}AC = E_r$.

理解①

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} & A & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = C \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = D \begin{pmatrix} y_1' \\ \vdots \\ y_m' \end{pmatrix}$$

$$\begin{pmatrix} y_1' \\ \vdots \\ y_m' \end{pmatrix} = \underbrace{D^{-1} A C}_{E_r.} \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$$

理解②.

$$\begin{tikzcd} V \arrow[r, "A"] & W \\ V' \arrow[u, "C", "\simeq"'] \arrow[r, "E_r."'] & W' \arrow[u, "D"', "\simeq"] \end{tikzcd}$$

① $\mathrm{Map}(A\times B, C) \xrightarrow{\sim} \mathrm{Map}(A, \mathrm{Map}(B, C))$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ f: A\times B \to C & & f': A \longrightarrow \mathrm{Map}(B, C) \\ f(-,-) & & a \to f(a,-) \end{array}$$

② A: set

V: vector space $= \mathrm{Map}(B, \mathbb{K})$

$\uparrow$ set

$\mathrm{Map}(A, \mathrm{Map}(B, K))$

$= \mathrm{Map}(A\times B, K)$

③ $V = \mathrm{Map}(A, K)$

$W = \mathrm{Map}(B, K)$

$\mathrm{LinMap}(V, W) = \mathrm{Map}(A, \underset{\mathrm{Map}(B, K)}{\underset{\parallel}{W}}) = \mathrm{Map}(A\times B, \mathbb{K})$

$\downarrow$

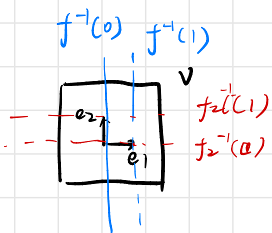
fix a basis vector in $V \Rightarrow W$

$\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \simeq \mathbb{R}^{n\times m}$

Example. of Rank Thm

$K^2 \xrightarrow{T_r} K^3$ rank $= 1$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

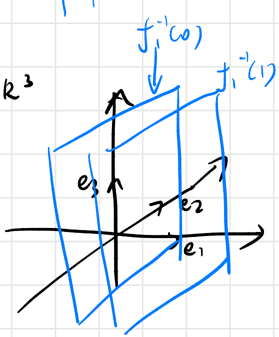


V^* has dual basis (f_1, f_2)

$f_1(e_1) = 1 \quad f_1(e_2) = 0$

$f_2(e_1) = 0 \quad f_2(e_2) = 1$

$V = \mathbb{R}^3$



$e_1 = (1, 0, 0)$
$e_2 = (0, 1, 0)$
$e_3 = (0, 0, 1)$

$V \to \mathbb{R}$

$f_1 = x$

$f_2 = y$

$f_3 = z$

Dual basis and adjoint map

- V v.s. $/\mathbb{K}$

dual space $V^* := \mathrm{Hom}(V, \mathbb{K})$ linear functions on V

e.g. 1 $\quad k = \mathbb{R} \quad V = \mathbb{R}$

$V^* \simeq \mathbb{R}$

$V^* = \{ f(x) \mid f(x) = a \cdot x \}$ $a \in \mathbb{R}$

$\simeq \mathbb{R}$

e.g. 2. $\quad V = \mathbb{R}^2$

$V^* = \{ f(x_1, x_2) \mid f(x_1, x_2) = a_1 x_1 + a_2 x_2,\ a_i \in \mathbb{R} \}$

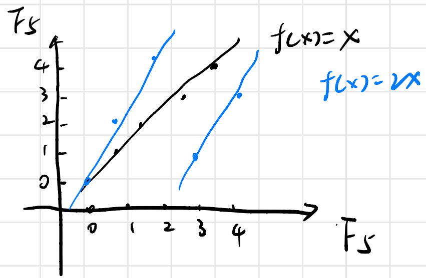
$\simeq \mathbb{R}^2$ (by extracting the coeff in front of x_1, x_2)

Ex 3. $\quad K = \mathbb{F}_5 \quad V = K^1$

$V^* = \mathrm{Hom}(V, \mathbb{F}_5)$

$= \mathrm{Lin\,Map}(\mathbb{F}_5, \mathbb{F}_5)$ $\leftarrow$ a linear map f is determined by where the basis vectors go

$= \{ f(x) \mid f(x) = a \cdot x \}$
$a \in \mathbb{K}$



Generalization:

- V: v.s/$\mathbb{K}$ of dim n
  $\{v_1, \cdots v_n\}$ be a basis

- V^* has a dual basis
  $\{f_1, \cdots, f_n\}$ defined by $(f_i, v_j) = \delta_{ij}$

$\langle -, - \rangle : V \times V^* \to K$
$(v, f) \mapsto f(v)$

$f_i(V) = f_i(a_1v_1 + \cdots a_nv_n)$
$= a_1 f_i v_1 + a_2 a_i v_2 + \cdots a_n f_i v_n$
$= a_i$

Pf: given any linear function
$f: V \to \mathbb{K}$

let $a_1 = f(v_1)$
$\vdots$
$a_n = f(v_n)$

Let $h = f - (a_1 f_1 + \cdots + a_n f_n)$ can be expressed as linear combination

then $h(v_1) = f(v_1) - (a_1 f_1(v_1) + a_2 f_2(v_1) + \cdots + a_n f_n(v_1))$
$= a_1 - (a_1 \cdot 1 + \underbrace{a_2 \cdot 0 + \cdots 0}_{0})$
$= 0$

Similarly $h(v_i) = 0$

$f_i(a_1v_1 + \cdots + a_nv_n) = a_i$
f_i depends not only on v_i but the entire basis $\{v_1 \cdots v_n\}$

Ex. V, $b = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$

V^*. $f_1^*(v_1) = 1$ $\quad f_1^*\left[2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = 2f^*\begin{pmatrix} 1 \\ 0 \end{pmatrix} + f^*\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$= 1$

$f_1^*(v_2) = 0$ $\quad f^*\left(3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 0$

$\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{pmatrix} f^*\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ f^*\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

adjoint map: given $T: V \to W$

we have adjoint $T^*: \underset{\substack{\Downarrow \\ g}}{W^*} \to V^*$ $\quad (\nearrow T^t)$

given $g \in W^*$ we need to define $\overbrace{T^*(g)}^{\text{in } V^*} = g \circ T : V \to \mathbb{K}$

(pull back of a function)

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "T^*(g) = g\circ T"'] & W \arrow[d, "g"] \\ & K \end{tikzcd}$$

$T^*(g(x)) = g(T(x))$

Suppose V has basis $\{v_1 \cdots v_n\}$

$\rightsquigarrow V^*$ has dual basis $\{v_1^*, - v^*_n\}$

similarly, $\{w_i\}$ basis of W

$\{w^*\}$ dual basis of W^*

$V = a_1 v_1 + \cdots + a_n v_n$

$= \langle v, v_1^* \rangle v_1 + \cdots + \langle v, v_n^* \rangle v_n$

$= \sum_{i=1}^{n} \langle v, v_i^* \rangle v_i$

$T: V \to W \quad v \in V,\ f \in W^*$

$\langle Tv, f \rangle = \langle v, T^* f \rangle$

because

$$\begin{tikzcd} & K \\ V \arrow[ur, "(f\circ T)v = T^*f"] \arrow[r, "T"'] & W \arrow[u, "f(T(v))"'] \end{tikzcd}$$

$(V^*)^* = V$ in finite dim cases

suppose $T_{ij} = \langle \underbrace{Tv_i}_{\in W}, \underbrace{w_j^*}_{\in W^*} \rangle$ $\iff$

pairing

$T(v_1) = T_{11} w_1 + \cdots + T_{1m} w_m$

$\vdots$

$T(v_n) = T_{n1} w_1 + \cdots + T_{nm} w_m$

then we can claim:

$\underbrace{T^*(\underbrace{w_j^*}_{W^*})}_{V^*} = \langle \underbrace{T^* w_j^*}_{V^*}, \underbrace{v_1}_{V} \rangle v_1^* + \cdots + \langle T^* w_j^*, v_n \rangle v_n^*$

$= \langle w_j^*, Tv_1 \rangle v_1^* + \cdots + \langle w_j^*, Tv_n \rangle v_n^*$

$= T_{1j} v_1^* + \cdots + T_{nj} v_n^*$

- The matrix for $T^*: W^* \to V^*$ in the dual basis, is the transpose of the matrix for $T: V \to W$

Cor: $\operatorname{rank}(T) = \operatorname{rank}(T^*)$

pf: By rank theorem, we can chose nice basis of V, W so that $T \rightsquigarrow \left(\begin{array}{c|c} \ddots & 0 \\ \hline 0 & 0 \end{array}\right) = E_r$

thus $T^* \rightsquigarrow E_r^t$ which has r non-zero entries

$\Rightarrow T^*$ has rank r

$$T^*(w_i^*) = \begin{cases} v_i^* & 1 \le i \le r \\ 0 & \text{else} \end{cases}$$

- $T: V \to W$

given a basis of V, W

set a matrix $[T] = \begin{pmatrix} T_{11} \cdots T_{1n} \\ \cdots \end{pmatrix}$

$K^n \longrightarrow K^m$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \underset{m \times n}{[T]} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

consider T as transforming stand basis vector to columns in T

$\operatorname{rank}(T) = \dim(\operatorname{im}(T)) = \dim(\text{span of the column vectors in } [T])$

$\operatorname{rank}(T^*)$ $= \dim(\text{span of the row vectors in } [T])$

$\dim(\operatorname{span}\{v_1 \cdots v_n\} \text{ in } V)$

$=$ the size of a maximal linearly independent subset in $\{v_1 \cdots v_n\}$ $\begin{cases} \text{rows} \\ \text{columns} \end{cases}$

$=$ the largest size k s.t. $\exists$

Ex. $\operatorname{rank}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix}$

$= 2$.

Linear Equations. in abstract

Given $T: V \to W$

$\ker(T) \subset V$

$\operatorname{im}(T) \subset W$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d] & W \\ V/\ker T \arrow[r, "\sim"] & \operatorname{im}(T) \arrow[u] \end{tikzcd}$$

Given any $w \in W$

$$T^{-1}(w) = \begin{cases} \emptyset, & w \notin \operatorname{im}(T) \\ v + \ker(T), & w \in \operatorname{im}(T) \end{cases}$$

for some $v \in T^{-1}(w)$

Concretely, A: $m \times n$ matrix

$K^n \longrightarrow K^m$

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Solving equation $A \cdot \vec{x} = \vec{b}$ has two possibilities

$$\begin{cases} \text{No soln} & \vec{b} \in \operatorname{im}(T) \\ \text{a particular soln } \vec{x} + \ker(T). & \vec{b} \in \operatorname{im}(T) \end{cases}$$

$$\ker(T) = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \middle| \underset{n \times n}{[A]} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \right\}$$

∘ dimension of intersection

$\dim(V) = \dim(V^*)$

$\operatorname{rank}(T) = \operatorname{rank}(T^*)$

- $W \subset V$ subspace

  $W \longrightarrow V \longrightarrow V/W$

  $\dim(V) = \dim(W) + \dim(V/W)$

- $V_1 \oplus V_2$

  $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$

$$\begin{aligned} \operatorname{rank}(T^*) &= \dim(\operatorname{im}(T^*)) \\ &= \dim((V/\ker T)^*) \\ &= \dim(V/\ker T) \\ &= \dim(\operatorname{im} T) \\ &= \operatorname{rank}(T) \end{aligned}$$

V v.s.

$V_1, V_2 \subset V$ subspace.

$V_1 + V_2 = \{ v_1 + v_2 \mid v_1 \in V_1, v_2 \in V_2 \} \subset V$

$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$

proof by constructing basis

$\{a_1 \cdots a_r, b_1 \cdots b_s, c_1 \cdots c_t\}$ $\{a_1 \cdots a_r, b_1 \cdots b_s\}$ $\{a_1 \cdots a_r, c_1 \cdots c_t\}$ $\{a_1, \cdots a_r\}$

A: $m \times n$ matrix

$Ax = b$

consistent $\iff b \in \text{colspace}(A)$
- unique soln $\Rightarrow \text{rank}(A) = n$.
- infinite soln $\Rightarrow \text{rank}(A) < n$.

$\iff \text{rank}(A) = m$

inconsistent $\iff$ rref of augmented m contains row $[0, 0, \ldots 0 \mid 1]$

$\Rightarrow \text{rank}\, A < m$

$\text{rank}(A) = \dim(\text{image}\, A) = \dim(\text{colspace}\, A) = \min(m, n)$

$= \dim(\text{row space}\, A)$

$= \#$ leading 1's in rref

$\#$ free variables $+$ $\#$ leading variables $= \#$ total variables

$\dim(\text{null}\, A)$ nullity $\qquad \text{rank}(A). \qquad n.$

$\text{rank}(A)$
- $< n \Rightarrow$
  - infinite.
  - no solution.
- $= n \Rightarrow$
  - unique.
  - no solution.

at most one solution $\iff$ column vectors are linearly independent

$\iff \text{rank}(A) = n$.

3.2. Gaussian elimination

Warm-up

- Swapping matrix: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ d & b \end{pmatrix} = \begin{pmatrix} d & b \\ a & c \end{pmatrix}$ Left multiplication by some "operator" $\Longleftrightarrow$ row operation

$\underline{\ \ } \ (\equiv) = (-)$

- $R_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & c \\ d & b \end{pmatrix} = \begin{pmatrix} a+d & b+c \\ d & b \end{pmatrix}$

- $R_{21} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & c \\ d & b \end{pmatrix} = \begin{pmatrix} a & c \\ a+d & b+c \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & c \\ d & b \end{pmatrix} = \begin{pmatrix} a+2d & c+2b \\ d & b \end{pmatrix}$$

$R_{12}(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ $\quad \Rightarrow R_{12}(\lambda)\, R_{12}(\mu) = R_{12}(\lambda+\mu)$

$R_{21}(\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ $\quad R_{12}(1)\, R_{12}(-1) = R_{12}(0) = Id.$

n unknown variables $x_1 \cdots x_n$

m equations

$$\begin{cases} a_{11}x_1 + \cdots a_{1n}\cdot x_n = b_1 \\ a_{21}x_1 \quad\quad + a_{2n}\cdot x_n = b_2 \\ \quad \vdots \quad\quad \vdots \quad\quad\quad \vdots \\ a_{m1}x_1 + \cdots \quad a_{mn}x_n = b_m \end{cases}$$

Concepts:

- extended matrix

$$m\left\{\left[\,\underbrace{A}_{n}\;\middle|\;\underbrace{b}_{1}\,\right]\right.$$

- row echelon form $\left[\begin{array}{ccc|c} 1 & & * & * \\ & \ddots & * & * \\ 0 & & \ddots & * \end{array}\right]$

① each row begins with a bunch of 0 and 1

$$\begin{pmatrix} 0 & 0 & 1 & \cdots & & \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ & & & & & \end{pmatrix} \begin{array}{l} p_1 = 3 \text{ pivot position in a row}_i \text{ is } p_i \\ p_2 = 5 \\ \vdots \end{array}$$

② the next row's $p_{i+1} > p_i$

- Reduced row echelon form:

use pivots to clean up the column above them

$$\begin{pmatrix} 0 & 1 & * & 0 & 0 & * & \vdots \\ 0 & 0 & 0 & 1 & 0 & * & \vdots \\ 0 & 0 & 0 & 0 & 1 & * & \\ \vdots & \vdots & & & & & \end{pmatrix}$$

- the case with no sol'n

if in the end, you want to find $(x_1, \cdots x_n)$ st.

$$\left(\begin{array}{ccc|c} 1 & \times & \times & \times \\ 0 & \ddots & & \times \\ & & a & \times \\ & & 0 & a \\ 0 & & 0 & 0 \end{array}\right)$$

- the case with more than 1 solution

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$\left(\begin{array}{cccccc|c} 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * \\ \vdots & \vdots & \vdots & \vdots & 1 & * & \\ 0 & 0 & 0 & \vdots & \vdots & \vdots & * \end{array}\right)$$

free variables: x_1 x_3 x_6

Given arbitrary x_1, x_3, x_6

decide: $\begin{cases} x_2 = b_1 - *x_3 - *x_6 \\ x_4 = b_4 - *x_6 \\ x_5 \cdots \end{cases}$

- Gauss elimination

$$[A \mid B] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \;\middle|\; A^{-1}B\right]$$

if A is $n\times n$ invertible matrix

$$[A \mid I_n] \rightsquigarrow [I \mid A^{-1}]$$

For $n \geq 1$

any $1 \le i \le j \le n$

let

$$\begin{array}{cl} S_{ij} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & 1 & & \\ & & 1 & & 0 & \\ & & & & & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \end{array} \\ \begin{array}{cc} \quad\quad\quad\quad\uparrow & \uparrow \\ \quad\quad\quad\quad i & j \end{array} & \end{array}$$

For $1 \le i \ne j \le n$, $\alpha \in K$

$$R_{ij}(\alpha) = \begin{bmatrix} 1 & & \overset{j}{\downarrow} & 0 \\ & \ddots & \alpha & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix} \qquad R_i(\alpha) = \begin{pmatrix} 1 & & \overset{i}{\downarrow} & 0 \\ & \ddots & & \\ & & \alpha & \\ 0 & & & \ddots \; 1 \end{pmatrix} \leftarrow i \quad \alpha \in K^*$$

Notation: $\{S_{ij}, R_{ij}(\alpha)\}$ are elementary matrices

Thm: Any invertible $n\times n$ matrix $A \in \underset{\text{general linear group}}{GL(n,k)}$ can be written as a finite product of elementary matrices

pf: let $A \overset{\text{row op}}{\rightsquigarrow} A_1 \overset{\text{row op}}{\rightsquigarrow} A_2 \rightsquigarrow \cdots \; A_N = I_n$.

be the history of $\underline{\underline{a}}$ Gauss Elimination.

$A_{i+1} = P_{i+1} A_i$

$\quad\quad\quad \hookrightarrow$ some elementary matrix

$$I = A_N = P_N\cdot(A_{N-1}) = P_N \cdot P_{N-1}\cdot A_{N-2} = (P_N \cdots P_1)A$$

$$P_1^{-1} \cdots P_{N-1}^{-1}\cdot P_N^{-1} = A$$

$$(S_{ij})^{-1} = S_{ij}$$

$$R_{ij}(\alpha)^{-1} = R_{ij}(-\alpha)$$

$$\alpha \ne 0 \quad R_i(\alpha) = R_i\left(\tfrac{1}{\alpha}\right)$$

Summary:

elementary row operation $\Longleftrightarrow$ multiply elementary matrices on the left

(1). Swap i,j rows of A $\Longleftrightarrow$ $S_{ij}\cdot A$

(2) multiplying row i by α $\Longleftrightarrow$ $R_i(\alpha)\cdot A$.

(3) add row j to row i $\Longleftrightarrow$ $R_{ij}(1)\cdot A$.

rank of row echelon form = # pivots = # non-zero rows

rank of a matrix = dim (column space / image)

= # lin. indep columns in A.

- LPU decomposition

Let M be a $n \times n$ invertible matrix

M can be written as $M = L \cdot P \cdot U$

L: $\begin{pmatrix} \times & & 0 \\ \times & \times & \\ \times & \times & \times \end{pmatrix}$

U: $\begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

P: permutation matrix

- given a bijection $\sigma = \{1, 2 \cdots n\} \longrightarrow \{1, 2, \cdots n\}$

i.e. $\sigma \in S_n$

$$(P_\sigma)_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{else.} \end{cases}$$

e.g. $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\quad P_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\Rightarrow$ left multiplication permutes rows

right $\sim \longrightarrow$ columns

- The set of $n \times n$ invertible matrices, denoted as $GL(n)$ "G"
  forms a group

  ① $\forall g_1, g_2 \in G$, $g_1 \cdot g_2 \in G$
  $$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$$

  ② $\exists e \in G$ identity element
  $$e g = g e = g$$

  ③ $\forall g \in G$, $\exists g^{-1} \in G$ s.t.
  $$g \cdot g^{-1} = g^{-1} \cdot g = e$$

- subgroup

  let G be a group, a subset $H \subset G$ is a subgroup if H satisfies ①②③

  Ex: let B_- denote the set of lower triangular matrices
  $$\begin{pmatrix} x & & 0 \\ & x & \\ x & & x \end{pmatrix}$$ satisfies ①②③.

  ③ e.g. $\begin{pmatrix} 1 & \\ a & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \\ -a & 1 \end{pmatrix}$

On finding inverse.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 5 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_2 = r_2 - 2r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 3 & 5 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_3 = r_3 - 3r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & 1 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_3 = r_3 - 5r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & -5 & 1 \end{array}\right)$$

$\therefore$ inverse of lower triangular is still inverse.

③

① Proof: Given lower triangular matrix, modify $r_i = r_i + c r_j$ for some $j < i$

then the new matrix is still lower triangular

$$L \to \begin{array}{c c} & j \\ i \to & \begin{pmatrix} 1 & & & \\ & 1 & & \\ c & & \ddots & \\ & & & 1 \end{pmatrix} L \end{array}$$

Similarly, the set of all invertible upper triangular matrices called B_+ forms a subgroup of $GL(n)$

In general, given subgroups $H_1, H_2 \subset G$ we can consider the subset

$$H_1 H_2 = \left\{ h_1 \cdot h_2 \;\middle|\; \begin{matrix} h_1 \in H_1 \\ h_2 \in H_2 \end{matrix} \right\} \subset G$$

- $G = GL(n)$

$B_- \cdot B_+ \subset G$

$\begin{pmatrix} \ast & 0 \\ \ast & \ast \end{pmatrix} \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix}$ covers all G?

e.g. $n=2$ $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} - & 0 \\ - & - \end{pmatrix} \begin{pmatrix} - & - \\ 0 & - \end{pmatrix}$

$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & - \\ - & be+cf \end{pmatrix}$

Given an arbitrary 2×2 invertible matrix

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

we use B_- left multiplication to simplify it

i.e. row reducing.

$\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} \cdot M$ chang the 1st row by factor a_1

$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ cr_1 + r_2 \end{pmatrix}$

① if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a \neq 0$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{r_1 = \frac{1}{a} r_1} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \xrightarrow{r_2 = r_2 - c r_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{bc}{a} \end{pmatrix}$

$d - \frac{bc}{a} \neq 0$.

$\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$

② if $a = 0$ with left multiplication of B^-

we have. $M = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 1 & d/c \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}$

$\therefore$ $G = B_1 \cdot P \cdot B_+$

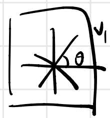
- Flag variety.
  - a complete flag in an n-dim v.s. V.
    is an increasing sequence of subspaces
    $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n$
    $\dim V_k = k$

Ex. $V = \mathbb{R}^2$

$V_0 \subset V_1 \subset V_2$
$\quad \| \qquad\quad \|$
$\quad 0 \qquad\quad V$

Let Fl_n be the set of flags in the vector space.

Ex: $Fl_2 = \mathbb{RP}^1$
Characterize V_1: with θ

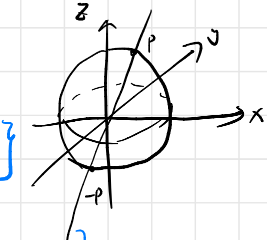


$\theta \in [0,\pi]/(0\sim\pi) \simeq [0,\pi)$
$= \mathbb{R}/\mathbb{Z}\cdot\pi \simeq \bigcirc = \mathbb{RP}^1$ projective
↑ 1 dim manifold

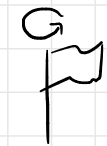
$Fl_n = Fl(n, (1,2,\cdots,n-1))$
$= \{V_1 \subset \cdots V_{n-1} \mid \dim V_k = k\}$
$Fl(n,(1,3)) = \{V_1 \subset V_3 \subset \mathbb{R}^n \mid \dim V_1 = 1\}$

Ex: $V = \mathbb{R}^3$ $\quad Fl_3(\mathbb{R}^3)$

V_1: specifying a line in $\mathbb{R}^3$ $\qquad S^2/\{\pm\}$
↓
sphere
$S^n = \{x_0^2 + \cdots + x_n^2 = 1\}$
$\subseteq \mathbb{R}^{n+1}$

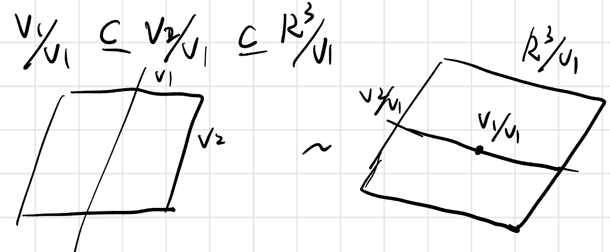


V_2: analogy to



- To parameterize $V_1 \subseteq \mathbb{R}^3$, we need a point $S^2/\{\pm\}$
  data of $\mathbb{RP}^2$.
- To parametrize V_2, is equivalent to specify data of a $\mathbb{RP}^1$

$V_1/V_1 \subseteq V_2/V_1 \subseteq \mathbb{R}^3/V_1$



$$Fl_{2,1} \xhookrightarrow{p^1} Fl_3 \qquad [V_0 \subset V_1 \subset V_2 \subset V_3]$$

$$\downarrow p^2 \qquad\qquad \downarrow$$

$$[V_0 \subset V_1 \subset V_3]$$

Summary: $Fl_n \xrightarrow{\text{map}} \mathbb{R}P^{n-1} \left(\{ V_1 \subset \mathbb{R}^n \mid \dim V_1 = 1 \} \right)$

$$Fl_n = \{ [V_0 \subset V_1 \subset \cdots V_n] \mid \dim V_i = i \}$$

$$\downarrow \cong$$

$$P^{n-1} = \{ [V_0 \subset V_1 \subset V_n] \mid \cdots \}$$

Numerically. given basis of V $(e_1 \cdots e_n)$

then we can produce a flag

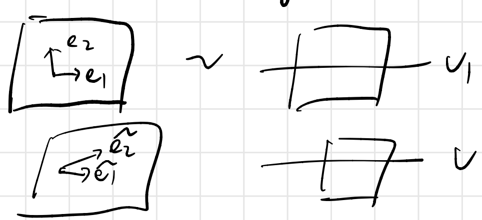
$V_1 = \mathrm{Span}\langle e_1 \rangle$

$V_2 = \mathrm{span}\langle e_1 \cdots e_2 \rangle$

$\vdots$

But redundant: different basis can produce the same flag

Ex: $V = \mathbb{R}^2$



(e_1, e_2) and $(\tilde{e}_1, \tilde{e}_2)$ generate the same flag

if $\tilde{e}_1 = a \cdot e_1$
$\tilde{e}_2 = b \cdot e_1 + c \cdot e_2$ $\Rightarrow (\tilde{e}_1, \tilde{e}_2) = (e_1, e_2) \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$\downarrow$

B_+

$$\therefore \quad Fl_n = GL(n) / B_+(n)$$

$$= \{ \text{the set of basis } e_1 \cdots e_n \} / (0\ \ast)$$

Two basis(es) are equivalent iff they differ by a multiplication of upper triangular matrix

$g \in GL_n \rightsquigarrow e_1, \cdots e_n$ as column vectors of g

$g \sim \tilde{g}$ if $g = \tilde{g} \cdot b_+$

Supplement:

linear map: $f: V \to W$

s.t. $f(u+v) = f(u) + f(w)$

$f(cu) = c f(u)$

linear form: $V \to F = \mathrm{Hom}(V, F)$.

e.g. V^*

bilinear form $B: V \times V \to F$.

B

$Q(x,y) = \sum_{1 \le i \le j \le n} q_{ij} x_i y_j$

s.t. $f(u_1+u_2, w) = f(u_1, w) + f(u_2, w)$

$\vdots$

$B(v, w) = v^T \hat{B} w$

satisfies axioms cuz distributive laws & the ability to pull out a scalar in matrix multiplication

basis $\{b_i\}$ in V $\quad v = \sum_i v_i b_i \quad w = \sum_i w_i b_i$

Quadratic form

$Q(x) = Q(x, x) = \sum_{1 \le i \le j \le n} q_{ij} x_i x_j$.

$[Q]$ $\quad q_{ji} = q_{ij}$ when $j \le i$

$\{f_1 \cdots f_n\}$ basis

$q_{ij} = Q(f_i, f_j) = Q(f_j, f_i) = Q_{ji}$

a basis is Q-orth if $Q(f_i, f_j) = 0$. $\forall i \neq j$

Lemma: Every quadratic form in K^n has an orthogonal basis.

Proof: Induction + Gram-Schmidt

$n=1$ $\quad Q(x) = qx$ $\quad$ trivial

3. Inertia Thm

Motivation:

- inner product on a 2-dim $\mathbb{R}$ space
  - $V = \mathbb{R}^2$
  - define quadratic form / symetric bilinear form

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_Q := (x_1, x_2) \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}_{Q} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1y_1 + 2x_1y_2 + 2x_2y_1 + x_2y_2$$

We can ask: for a given vector $\vec{v} \in V$,
what is the set of all vectors "perpendicular" to $\vec{v}$,

$$\{ w \in V \mid \langle v, w \rangle = 0 \} := \vec{v}^{\perp Q}$$

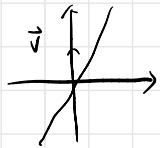
since the set satisfies linearity $\Rightarrow$ linear subspace

Eg.

$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad w = (x_1, x_2)$ $\quad (x_1, x_2)\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$

$(x_1, x_2)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0.$

$x_1 + 2x_2 = 0.$



$\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow 2x_1 + x_2 = 0.$

- Let $\mathbb{K}$ be a field
  V be a fin. dim. vector space$/\mathbb{K}$

- Let $B: V \times V \to \mathbb{K}$
  be a symetric bilinear form
  $B(v,w) = B(w,v) \in \mathbb{K}$

- B is equivalent to quadratic form
  $Q: V \to \mathbb{K}$.

by setting $v=w \Rightarrow Q(v) = B(v,v)$

conversely $B(v,w) = \frac{1}{4}\left(B(v+w, v+w) - B(v-w, v-w)\right)$

or

$$= \frac{1}{2}\left(B(v+v, v+w) - B(v,v) - B(w,w)\right)$$

$$= \frac{1}{2}\left(Q(v+w) - Q(v) - Q(w)\right)$$

$$= \frac{1}{2}\left[(a+b)^2 - a^2 - b^2\right] = ab.$$

$\left(\begin{array}{l} (a+b)^2 = a^2+b^2+2ab \\ (a-b)^2 = a^2+b^2-2ab. \end{array}\right)$ behave like numbers.

Given a basis $e_1 \cdots e_n$ of V and a bilinear form B.
we can produce a matrix.

$$B_{ij} = B(e_i, e_j)$$

B symetric $\Rightarrow B(e_i, e_j) = B(e_j, e_i) \Rightarrow B_{ij} = B_{ji}$

If $v = v_1e_1 + \cdots + v_ne_n$ $v_i \in \mathbb{K}$
$w = w_1e_1 + \cdots w_ne_n$ $w_i \in \mathbb{K}$.

推导 unclear

then $B(v,w) = B\left(\sum_{i=1}^{n} v_ie_i, \sum_{j=1}^{n} w_je_j\right)$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} B(v_ie_i, w_je_j)$$ why j rather than i

$$= \sum_{i,j} v_iw_jB_{ij}$$

$$= (v_1 \cdots v_n)\begin{pmatrix} B_{11} & B_{12} & \cdots \\ & & \end{pmatrix}\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

Side; example:

$$B(a\vec{v}+b\vec{w}, c\vec{m}+d\vec{o})$$

$$= B(a\vec{v}, c\vec{m}) + B(a\vec{m}+d\vec{o}) + B \cdots$$

$$= acB(\vec{v}, \vec{m}) + adB(\vec{m}, \vec{o})$$

$bc + bd \cdots$

- A basis $\{e_i\}_{i=1}^n$ of V is orthogonal w.r.t. bilinear form B if for $i \neq j$, we have $B(e_i, e_j) = 0$

i.e. $[B_{ij}]$ is a diagonal matrix $\to$ $B_{ij} = B_{ji}$, $[B] = [B]^t$

Lemma: every bilinear form B on a finite dim V admits an orthogonal basis.

$\Leftrightarrow$ $\exists$ a basis $\{e_1 \cdots e_n\}$ of V s.t. $e_i \perp_B e_j$ w.r.t. to B. $B(e_i, e_j) = 0$ if $i \neq j$

Pf: Induction:

- if $B \equiv 0$, then any basis works.
- On $\dim_K V$ Suppose the statement is true for any v.s V, B, $\dim_K V \leq n-1$
- Consider any v.s. V and a bilinear form B on V s.t. $\dim V = n$

  if $B \equiv 0$ ✓

  if $B \neq 0$ then $\exists$ a vector $v \in V$ s.t. $Q(v) = B(v,v) \neq 0$

Let $W = \vec{v}^{\perp_B} = \{w \in V \mid B(w,v) = 0\}$ then $v \notin W$

Claim: $\dim W = n-1$

because consider: $T: V \to K$, $w \mapsto B(w,v)$

$W = \ker(T)$

By Rank-Nullity Thm:

$\dim V = \dim(\operatorname{im}(T)) + \dim(\ker(T))$

and $\operatorname{im}(T) = K$ ($\because T(v) \neq 0$)

$\therefore \dim(\operatorname{im} T) = 1$

$\therefore \dim(\ker T) = n-1$

By induction, for $(W, B|_{W \times W})$ or (W, B) the statement holds. Thus, $\exists$ a basis of W, $\{e_1 \cdots e_{n-1}\}$, that is orthogonal w.r.t. B

$B: V \times V \to K$.

Given: $f: A \to B$ map of sets, $A' \subset A$ subset

$f|_{A'}: A' \to B$

let $e_n = v$ then

$\{\underbrace{e_1, \cdots e_{n-1}}_{W}, e_n\}$ forms a basis of V

let $\{e_1 \cdots e_n\}$ be an orthognal basis of B

$$[B_{ij}] = \begin{bmatrix} B_{11} & & 0 \\ & B_{22} & \\ 0 & & \ddots & B_{nn} \end{bmatrix}$$

- If $K = \mathbb{R}$ then we can reorder the basis s.t.

$\underbrace{B_{11}, \cdots B_{pp}}_{p \text{ many}} > 0$, $\underbrace{B_{p+1} \cdots B_{q+p}}_{q \text{ many}} < 0, 0 \cdots$

$$[B_{ij}] = \begin{bmatrix} + & & & & & & \\ & + & & & & & \\ & & + & & & & \\ & & & - & & & \\ & & & & - & & \\ & & & & & - & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{bmatrix}$$

we call such B of signature (p, q)

By rescaling $\{e_i\}$, we can make

$$[B_{ij}] = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{bmatrix}$$

- If $K = \mathbb{C}$

Suppose $B(e_i, e_i) = \lambda_i \in \mathbb{C}$
$B(e_i, e_j) = 0$ for $i \neq j$

$$[B_{ij}] = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_r & \\ & & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

let $\tilde{e_i} = \frac{1}{\sqrt{\lambda_i}} e_i$ for $1 \le i \le r$

$\tilde{e}_i = e_i$ for $i > r$

$B \cdot (\tilde{e}_i, \tilde{e}_i) = \left(\frac{1}{\sqrt{\lambda_i}}\right)^2 \cdot \lambda_1 = 1$

Ex:

$$B_{ij} = \begin{pmatrix} \overbrace{1 \; \ddots \; 1}^{r} & \\ & 0 \; \ddots \; 0 \end{pmatrix}$$

$Q(x_1 e_1 + \cdots + x_n e_n) = x_1^2 + \cdots + x_r^2 + 0 \cdots 0$

Change Basis

Find orthogonal basis for a 3x3 matrix (MT)

- Let B be a bilinear form on V
- let $\{e_1 \cdots e_n\}$ be a basis of V

let $\{\tilde{e}_1, \cdots \tilde{e}_n\}$ another basis of V

we have $(e_1 \cdots e_n) = (\tilde{e}_1, \cdots \tilde{e}_n) \cdot A$

$(e_1 \cdots e_n) A^{-1} = (\tilde{e}_1, \cdots \tilde{e}_n)$

for some invertible matrix A

$$e_i = (\tilde{e}_1, \cdots \tilde{e}_n) \begin{pmatrix} A_{1i} \\ A_{2i} \\ \vdots \\ A_{ni} \end{pmatrix} = \sum_{j=1}^{n} \tilde{e}_j A_{ji}$$

$$\tilde{e}_i = (e_1, \cdots, e_n) \begin{pmatrix} (A^{-1})_{1i} \\ \vdots \\ (A^{-1})_{ni} \end{pmatrix} = \sum_{j=1}^{n} e_j (A^{-1})_{ji}$$

Ex: $B_{ij} = B(e_i, e_j)$

$\tilde{B}_{ij} = B(\tilde{e}_i, \tilde{e}_j)$

express $\{e_i\}$ using $\{\tilde{e}_i\}$. we have

$$B_{ij} = B(e_i, e_j) = B\left(\sum_{a=1}^{n} \tilde{e}_a A_{ai}, \sum_{b=1}^{n} \tilde{e}_b A_{bj}\right)$$

$$= \sum_{a,b} A_{ai} A_{bj} B(\tilde{e}_a, \tilde{e}_b).$$

$$= \sum_{a,b} A_{ai} A_{bj} \tilde{B}_{ab}$$

scalars.

$$= \sum_{a,b} \underbrace{(A^t)_{ia} \tilde{B}_{ab}} \underbrace{A_{bj}}$$

$$= [A^t \cdot \tilde{B} \cdot A]_{ij}$$

$$[B] = A^t \cdot [\tilde{B}] \cdot A$$

basic moves:

- Swap i & j rows
  i & j columns

  $B \rightsquigarrow S_{ij}^t B S_{ij}$ $\quad S_{ij} = \begin{array}{c} \begin{matrix} & & & i & j & \end{matrix} \\ \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & 1 \end{pmatrix} \end{array}$

- change ith row by adding a multiple of j-th row

AND change ith column ......

$r_i = r_i + \lambda r_j$

$c_i = c_i + \lambda \cdot c_j$.

$B \rightsquigarrow$

Ex.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$$

$r_2 = r_2 - 2r_1$

$c_2 = c_2 - 2c_1$

Gram-schmidt Thm.

$$\text{proj}_u(v) = \frac{\langle u, v\rangle}{\langle u, u\rangle} u.$$

$u_1 = v_1$

$u_2 = v_2 - \text{proj}_{u_1}(v_2)$

$u_3 = v_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(v_3)$

Adapted Gram-schmidt

Q- orthogonal basis -

$v_1 = u_1$

$$v_2 = u_2 - \frac{Q(u_2, v_1)}{Q(v_1, v_1)} v_1$$

$\vdots$

$$v_n = u_n - \sum_{1 \leq i \leq j} \frac{Q(v_i, v_i)}{Q(v_i, v_j)} \cdot v_i$$

Lemma & Gram-Schmidt.

- a bilinear form B on V is non-degenerate ~~an inner product~~
if for any $0 \neq v \in V$, $\exists\ \tilde{v} \in V$. s.t. $B(v, \tilde{v}) \neq 0$
$\Leftrightarrow$ $B(v, \_) : V \to K$ is non-zero.

- When we diagonalize a non-degen symetric bilinear form

$$[B] \mapsto [\tilde{B}] = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$\lambda_i \neq 0$

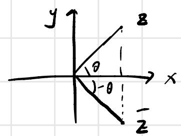
- If $K = \mathbb{R}$, an inner product on V is a bilinear form B s.t.
$B(v,v) > 0$ for all $v_{\neq 0} \in V$

i.e. in diagnoal form, one can find $[\tilde{B}] = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

Complex conjugation on $\mathbb{C}$

$z = x + i\cdot y$, $x, y \in \mathbb{R}$ $i \in \mathbb{C}$
$\bar{z} = x - iy$

$z = r\cdot e^{i\theta}$
$\bar{z} = r\cdot e^{-i\theta}$



$\underline{z\cdot\bar{z} = x^2+y^2}$ $\qquad$ $r\cdot e^{i\theta}\cdot r\cdot e^{-i\theta} = \underline{r^2} \geq 0$

non-negative $\mathbb{R}$

Generalize to vector space other than $\mathbb{C}$

- Let $S : V\times V \to \mathbb{C}$ s.t. " S is $\mathbb{C}$-linear in the 2nd slot" and $\mathbb{C}$-anti-linear in the 1st slot"

$\Longleftrightarrow$ $\quad S(v, aw_1 + bw_2) = a\cdot S(v, w_1) + b\cdot S(v, w_2)$
$\qquad S(aw_1 + bw_2, v) = \bar{a}\, S(w_1, v) + \bar{b}\, S(w_2, v)$

Ex: $\mathbb{C}^n$ $\qquad z_i \in \mathbb{C}$. $\quad$ we can define the std. ses-linear form

$V = (z_1 \cdots z_n) \in \mathbb{C}^n$
$W = (u_1 \cdots u_n) \in \mathbb{C}^n$

$S(v, w) = \bar{z}_1\cdot u_1 + \bar{z}_2\cdot u_2 + \cdots \bar{z}_n\cdot u_n$.

$S^*(x, y) := \overline{S(y, x)}$ $\quad$ Hermitian adjoint

$P = H + Q = S(v, w)$
$H = \frac{1}{2}(P + P^*) = \frac{1}{2}(S(v,w) + \overline{S(w,v)})$
$Q = \frac{1}{2}(P - P^*)$

matrix:

$P(x,y) = \sum_{i=1}^{m}\sum_{i=1}^{n} \bar{x}_i P_{ij} y_j \qquad P_{ij} = P(e_i, e_j)$

$P^*(x,y) = \overline{P(y\, x)} \quad \Rightarrow \quad P^*_{ij} = \overline{P_{ji}}$

$\Rightarrow \quad P^* = \overline{P^t}$

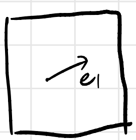
A sesquilinear form S is hermitian if $S(v,w)=\overline{S(w,v)}$

$$iS(v,w) = i\,\overline{S(w,v)} = -\overline{iS(w,v)}$$

Side note:

If V is a vector space over $\mathbb{C}$, $v \in V$, then it doesn't make sense to say $\bar{v}$.

let V be a 1-$\dim_{\mathbb{C}}$ v.s over $\mathbb{C}$



V.

let $e_1 \in V$ be a basis of V

then $v = c_1 \cdot e_1$, $c_1 \in \mathbb{C}$

suppose we define $\bar{v} = \bar{c}_1 e_1$

then this def depends on the choice of e_1

If $\tilde{e}_1 = e^{i\theta} \cdot e_1$ and $v = \tilde{c}_1 \cdot \tilde{e}_1$

$\Rightarrow$ $\bar{v} = \overline{\tilde{c}_1} \cdot \tilde{e}_1$ then we have contradiction.

Ex:

non-hermitian sesquilinear form: $V = \mathbb{C}$

$S: V \times V \to \mathbb{C}$

$(z,w) \mapsto e^{i\theta} \cdot \bar{z} \cdot w$

In general:

$$S(\vec{z},\vec{w}) = \sum_{i=1}^{n} c_i \cdot \bar{z}_i \cdot w_i \qquad c_i \in \mathbb{C}$$

$$S(\lambda\vec{z},\vec{w}) = \bar{\lambda}\, S(\vec{z},\vec{w})$$

$$S(\vec{z},\lambda\vec{w}) = \lambda\, S(\vec{z},\vec{w})$$

$$S(\vec{z},\vec{w}) = \sum_{i=1}^{n} c_i\, \bar{z}_i \cdot w_i$$

$$\overline{S(\vec{w},\vec{z})} = \overline{\sum_{i=1}^{n} c_i \cdot z_i \cdot \bar{w}_i} = \sum_{i=1}^{n} \bar{c}_i\, \bar{z}_i \cdot w_i$$

If S is a hermitian sesquilinear form

$S: V \times V \to C$

then knowing its value on $S(v,v)$ is enough to recover $S(v,w)$

Proof: Exercise

$$S(v,v) = \overline{S(v,v)} = a$$

$$\bar{c}_i\, \bar{v}_i\, \bar{v}_i = c_i v_i v_i$$

Let $H: V \times V \to \mathbb{C}$ be a hermitian form

$v, w \in H$

$v \perp w$ w.r.t. H if $H(v, w) = 0$.

$v \perp w \Leftrightarrow w \perp v$

Lemma: Given any hermitian form H,

$\exists$ an orthogonal basis $e_1 \cdots e_n$ of V.

i.e. $e_i \perp e_j \ \forall i \neq j$

$e_i \perp e_j \Leftrightarrow e_j \perp e_i$ is guaranteed by $S(v, w) = \overline{S(w, v)}$ symmetry

proof / algorithm:

- If $\dim V = 0$, then done
  If $H \equiv 0$, then pick any basis of V.

- Otherwise, pick a $v \in V$, s.t. $H(v) \neq 0$
  let $V' = v^{\perp} := \{w \in V \mid w \perp v\}$ (Imposing 1 equation $H(v, w) = 0$ lowers the dim by 1)
  repeat the process replacing V by V'

Let H be a Hermitian form on V, $\{e_1 \cdots e_n\}$ basis

we can define $H_{ij} := H(e_i, e_j) \in \mathbb{C}$

$[H_{ij}]$ $n \times n$ matrix

By def of hermitian $H_{ij} = \overline{H_{ji}}$

any complex matrix A, we can define $A^+ := \overline{A^t}$

if $A = A^+ = \overline{A^t}$, then A is a Hermitian matrix

$\because$ $S(v, v) = \overline{S(v,v)}$ complex conjugate of it selue.

$\Rightarrow$ $H_{ii} \in \mathbb{R}$

Ex: $[H_{ij}] = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$

Cor: Given H: hermitian form on V. $\exists$ basis $\{e_1 \cdots e_n\}$ of V

s.t. $H(e_i, e_j) = \begin{cases} 0 & i \neq j \\ +1/-1/0 & i = j \end{cases}$

Ex: Let $V = \mathbb{C}^2$, $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$[H] = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$

Q: find a new basis $(\tilde{e}_1, \tilde{e}_2)$ s.t. $[H_{ij}]$ is diagonalized

A: column operation: $C_2 \leftarrow C_2 - C_1 \cdot \frac{1+i}{2}$ $\begin{pmatrix} 2 & 0 \\ 1-i & 2 \end{pmatrix}$

$\downarrow$

Row operation: $r_2 \leftarrow r_2 - r_1 \cdot \overline{\left(\frac{1+i}{2}\right)}$ $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$[H_{ij}] \rightsquigarrow \overline{A^t}\,[\tilde{H}_{ij}]\cdot A$

$H: \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}$

$(z_1, z_2)\ (w_1, w_2) \longmapsto \overline{(z_1, z_2)} \begin{pmatrix} 2 & 1+i \\ 1-i & 4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

Recall:

- (complete) flag in a vector space V.

  Def: increasing sequence of vector subspaces.

  $0 = V_0 \subset V_1 \cdots V_n = V$ s.t. $\dim V := n$

- Given a basis $\{e_1 \cdots e_n\}$ of V

  $\to$ obtain a standard flag associated to the basis

  $V_1 = \operatorname{span}\{e_1\}$

  $V_2 = \operatorname{span}\{e_1, e_2\}$

  $\vdots$

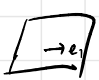
  $V_k = \operatorname{span}\{e_1 \cdots e_k\}$

- Given a flag $V_\bullet$ in V and a basis $\{e_i\}$, we say the basis is adapted to the flag

  if $V_k = \operatorname{span}\{e_1 \cdots e_k\}$

Lemma: Given a flag $V_\bullet$, $\exists$ a basis $\{e_i\}$ adapted to $V_\bullet$

Pf: 1. pick any $e_1 \subset V_1$ s.t. e_1 is a basis of V_1

2.  V_2 pick any vector $e_2 \in V_2$, $e_2 \notin V_1$

   s.t. $V_2 = \operatorname{span}\langle e_1, e_2 \rangle$.

$\vdots$

pick $e_k \in \underbrace{V_k \setminus V_{k-1}}_{\text{remove / subtract}}$

- Suppose we have certain notion of orthogonality:

Ex 1: - Given a symm bilinear form

$B: V \times V \to K$

we say $v \perp w$ if $B\langle v, w \rangle = 0$.

and $v \perp w$ iff $w \perp v$ by symmetricity

Ex 2: - Given a Hermitian form

$H: V \times V \to \mathbb{C}$

$H(v, w) = \overline{H(w, v)}$

$v \perp w$ if $H(v, w) = 0$

Then, given a flag $V_\bullet$, we can construct orthogonal basis adapted to $V_\bullet$

Pf: 1. pick any $e_1 \subset V_1$ s.t. e_1 is a basis of V_1

2. pick any vector $e_2 \in V_2$, $e_2 \notin V_1$

$\boxed{\text{AND s.t. } e_2 \perp e_1}$

$\uparrow$ Gramshmit 思想.

- Let $\tilde{e}_2 \in V_2 \setminus V_1$

suppose we are working with Hermitian form

if $H(\tilde{e}_2, e_1) \neq 0$

then we define $e_2 = \tilde{e}_2 - \lambda e_1$ s.t. $H(e_2, e_1) = 0$

$$\downarrow$$

$$H(\tilde{e}_2 - \lambda e_1, e_1) = 0$$

$$\Leftrightarrow \quad H(\tilde{e}_2, e_1) - \bar{\lambda} H(e_1, e_1) = 0.$$

$$\bar{\lambda} = \frac{H(\tilde{e}_2, e_1)}{H(e_1, e_1)_{\times}} \text{ works.}$$

what if $H(e_1, e_1) = 0$?

Therefore, we need to assume Hermitian form is positive definite.

修补的逻辑.

Def: Let V be a v.s./$\mathbb{C}$

$H: V \times V \to \mathbb{C}$ a hermitian form

H is positive definite if $\forall\, 0 \neq v \in V$, $H(v, v) > 0$

Prop: Given a positive def. Hermitian form H, and $\forall$ flag V.

$\exists$ orthogonal basis $\{e_i\}$ adapted to V.

Recall: A hermitian form H on $\mathbb{C}^n$ is equivalent to one of the following

$$H(z, z) = H(z) = \sum_{i=1}^{p} |z_i|^2 - \sum_{j=1}^{q} |z_{p+j}|^2$$

- the positive definite $H \Leftrightarrow H(z) = |z_1|^2 + \cdots |z_n|^2$

- if $p + q = n$, $p > 0$, $q > 0$

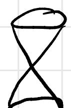
Ex: $\mathbb{C}^2$ $\quad H(z) = |z_1|^2 - |z_2|^2$

then there are $z = (z_1, z_2) \neq 0$ s.t. $H(z) = 0$.

e.g. $z = (1, 1)$ $\quad z = (1, i)$

Q: $Null(H) = \{ z \in \mathbb{C}^2 \mid |z_1|^2 - |z_2|^2 = 0 \}$ is it a vector space

No. It's a cone.



counterexample: $\left\{ \begin{array}{l} (1, 1) \\ (1, -1) \end{array} \right. \in Null(H)$

$(2, 0) \notin$

It's closed under scalar mul.

$\times$ for multiplication $\because$ absolute value is non-linear

Let H be a $n \times n$ Hermitian matrix

$$H = \begin{bmatrix} \boxed{H_{11}} & H_{12} & \cdots & \cdots \\ H_{21} & H_{22} & & \\ \vdots & & & \\ \vdots & & & H_{nn} \end{bmatrix}$$

$[H_{11}]$ 1×1 $\rightsquigarrow$ $\Delta_1 = \det(H_{11})$

$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ 2×2 $\rightsquigarrow$ Δ_2

Principal minors:

- $\Delta_k = \det$ (upper left $k \times k$ submatrix of H)

Recall: if H is a hermitian matrix i.e. $H = \overline{H}^t$

then $\det H = \det(\overline{H}^t) = \overline{\det(H^t)} = \overline{\det(H)}$

$\Rightarrow \det H = \mathbb{R}$

$\det(\overline{C}) = \overline{\det(C)}$

Assume that

$\Delta_k \neq 0$ for $k = 1, 2, \cdots n$

Thm: Let q be the number of sign changes in $\{\Delta_0, \Delta_1, \Delta_2, \Delta_3 \cdots, \Delta_n\}$ $\Delta_0 = 1$

then q = the negative inertia index of H

Ex: if H is a diagonal matrix

$$H = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & +1 & \\ & & & -1 \end{pmatrix}$$

negative index $= 2$

$\Delta_0 = 1 \mid \Delta_1 = 1 \mid \Delta_2 = -1 \mid \Delta_3 = -1 \mid \Delta_4 = +1 \mid$

$$H = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$\Delta_k = \lambda_1 \cdots \lambda_k$

if $\lambda_k < 0$, then Δ_{k-1} and Δ_k has a sign change

Let $V=\mathbb{C}^n$, $e_1\cdots e_n$ be the statd basis of $\mathbb{C}^n$

$V_\bullet$ be standard flag for $\{e_i\}$

- By Gram-Schmidt process, obtain a new orthognal basis $\{f_1\cdots f_n\}$

s.t. $\operatorname{span}(f_1)=\operatorname{span}(e_1) \quad = V_1$

$\operatorname{span}(f_1,f_2)=\operatorname{span}(e_1,e_2)=V_2$

$\vdots$

and $H(f_i,f_j)=0$ if $i\neq j$

$H(f_i,f_i)=\pm 1$

$D = C^\dagger H C \qquad C=\begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix} = \begin{bmatrix} f_1^\dagger \cdot H \cdot f_1 & f_1^\dagger \cdot H \cdot f_2 \\ & \end{bmatrix}$

$$= \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & \cdots \end{array} \\ \begin{pmatrix} \ast & \ast & \cdots \\ 0 & \ast & \cdots \\ 0 & 0 & \\ \vdots & 0 & \\ 0 & \vdots & \end{pmatrix} \end{array}$$

$$\begin{pmatrix} (\quad \overline{f_1}^t \quad) \\ (\quad \overline{f_2}^t \quad) \\ (\quad \overline{f_3}^t \quad) \end{pmatrix} \begin{pmatrix} & & \\ & H & \\ & & \end{pmatrix} \begin{pmatrix} \begin{pmatrix} f_1 \end{pmatrix} & \begin{pmatrix} f_2 \end{pmatrix} & \begin{pmatrix} f_3 \end{pmatrix} \end{pmatrix}$$

claim: $\Delta_k(D)$ and $\Delta_k(H)$ has the same sign

Let D_k be the upper left $k\times k$ matrix of D.

$H_k\cdots$ similary

$C_k\cdots$

$\det(D_k) = \det(C_k^\dagger)\cdot\det(H_k)\cdot\det(C_k)$

$= \overline{\det(C_k)}\cdot\det(C_k)\cdot\det(H_k)$

$= |\det(C_k)|^2\cdot\det(H_k)$

* Different context of diagonalization.

Given M, find invertible matrix A s.t.

$A^t M A$ is diagonal

or $\bar{A}^t M A$ ~

or $A^{-1} M A$ ~

• diagonalize a Hermitian form $H: V \times V \to \mathbb{C}$ means.

find basis $e_1 \cdots e_n$ s.t. $H_{ij} = H(e_i, e_j)$ is diagonalized

$$[\tilde{e}_i] = C[e_i]$$

$$[e_i] v = [\tilde{e}_i] \cdot \tilde{v} \qquad v = C \cdot \tilde{v}$$

$$w = C \cdot \tilde{w}$$

$$\tilde{Q} = C^t Q C.$$

$$Q(v, w) = v^t Q w$$

$$= \tilde{v}^t \cdot C^t Q \cdot C \cdot \tilde{w}$$

Ex.

$$Q = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$Q_0 = Q$.

$$C_0^t \cdot Q_0 \cdot C_0 = \begin{pmatrix} 9_1 & 0 \\ 0 & \boxed{} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{C_0} \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad 2x_1 x_2.$$

$$\tilde{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$(1, 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\tilde{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\tilde{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Chap 4. Eigenvalue Problem:

General definition: Let V be a vector space$/K$

Let $T: V \to V$ be a linear map

$v \in V$ is an eigenvector of T, with eigen value $\lambda \in K$,

if $T(v) = \lambda \cdot v$

$T \in \mathrm{End}(V) = \mathrm{Hom}(V, V)$

↑ Endomorphism

Dream Case:

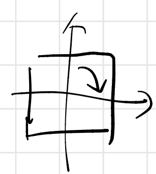
Give $T: V \to V$, $\exists$ a basis of V $\{e_1 \cdots e_n\}$ and numbers $\lambda_1 \cdots \lambda_n$ s.t.

$Te_i = \lambda e_i$

Dream may not be true: (minor cases)

Ex 1. $V = \mathbb{R}^2$, $k = \mathbb{R}$ $\quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

This can't be diagnolized.



$T(v) =$ rotate cw $90°$.

X in the same directon

Ex 2: $k = \mathbb{C}$ $\quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

one can have one eigenvector $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $T(e_1) = 1 \cdot e_1$

X find 2nd one.

fix e_1, move e_2

shearing.

Hermitian vector space:

- V: v.s/$\mathbb{C}$. $H: V \times V \to \mathbb{C}$ positive definite Hermitian form.
- $\|v\|^2 = H(v,v)$ $\quad \|\times\|: V \to \mathbb{R}_{\geq 0}$

tuple. $(V, \langle -, - \rangle)$

complex vector space with positive definite hermitian form on V.

Cauchy - Schwarz ineq $\Rightarrow$ triangle ineq

Notation: V: n-dim complex v.s.

$\langle -, - \rangle : V \times V \to \mathbb{C}$ Hermitian form

$\langle \lambda v, \mu w \rangle = \bar{\lambda} \mu \langle v, w \rangle$ $\quad \mu, \lambda \in \mathbb{C}$, $v, w \in V$

positive definite: $v \neq 0 \iff \langle v, w \rangle > 0$.

$|v| = \sqrt{\langle v, v \rangle}$ "length of v" "norm of v"

C.S ineq: Given $\forall v, w \in V$,

$$|\langle v, w \rangle|^2 \leq |v|^2 \cdot |w|^2$$

Pf: $|v| = 0$, or $|w| = 0$, then $\langle v, w \rangle = 0$ ✓

If $|v|, |w| \neq 0$. we can define

$\hat{v} = v/|v|$ $\quad \hat{w} = w/|w|$ (length 1)

$$|\langle v, w \rangle|^2 \leq |v|^2 \cdot |w|^2$$

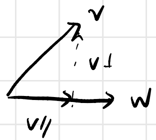
$$\iff \left| \frac{\langle v, w \rangle}{|v| \cdot |w|} \right|^2 \leq 1$$

$|v|, |w| \in \mathbb{R}$ $\quad \iff \left| \left\langle \frac{v}{|v|}, \frac{w}{|w|} \right\rangle \right|^2 \leq 1$

$$\iff |\langle \hat{v}, \hat{w} \rangle| \leq 1$$

Simplify notation: $\tilde{v} \rightsquigarrow v$ of $L(l)$

$\tilde{w} \longrightarrow w$.



$v = v_\parallel + v_\perp$ $\mathbb{C}$-linear in v.

$v_\parallel = \langle w, v\rangle w$

$v_\perp = v - \langle w, v\rangle w$

Indeed, $\langle w, v_\perp\rangle = \langle w, v - \langle w, v\rangle w\rangle$

$= \langle w, v\rangle - \langle w, v\rangle \langle w, w\rangle$

$= 0$ $\Rightarrow |w| = 1$

set $\langle w, v\rangle = \lambda$

$0 \leq \langle v_\perp, v_\perp\rangle$

$= \langle v_\perp, v - \lambda w\rangle$

$= \langle v_\perp, v\rangle - \lambda \langle v_\perp, w\rangle$

$= \langle v - \lambda w, v\rangle$

$= \langle v, v\rangle - \langle \lambda w, v\rangle$

$= 1 - \bar{\lambda} \cdot \langle w, v\rangle$

$= 1 - \bar{\lambda} \cdot \lambda = 1 - |\lambda|^2$

$\Rightarrow |\lambda|^2 \leq 1$

$\Rightarrow |\langle w, v\rangle|^2 \leq 1$ for any v, w

with $|w| = 1$ $|v| = 1$

Triangle inequality

$|v + w| \leq |v| + |w|$

$\Leftrightarrow |v + w|^2 \leq (|v| + |w|)^2$

$\langle v+w, v+w\rangle$ $\Leftrightarrow |v|^2 + |w|^2 + \underbrace{\langle v, w\rangle + \langle w, v\rangle}_{2 \cdot \mathrm{Re}\langle v, w\rangle} \leq |v|^2 + |w|^2 + 2|v| \cdot |w|$

$\Leftrightarrow \mathrm{Re}(\langle v, w\rangle) \leq |v| \cdot |w|$

$\Leftarrow$ $|\langle v, w\rangle| \leq |v| \cdot |w|$

let $V \langle -,- \rangle$ be a Hermitian v.s.

$\Rightarrow$ $\exists$ ONB $\{e_1 \cdots e_n\}$

$\Rightarrow$ $\exists$ isomorphism between

$$(V, \langle , \rangle) \xrightarrow{\sim} (\mathbb{C}^n, \langle -,- \rangle_{std})$$

$$\langle z, w \rangle_{std} = \sum_{i=1}^{n} z_i \overline{w_i}$$

$z = (z_1 \cdots z_n)$
$w = (w_1 \cdots w_n)$

$$v = v_1 e_1 + \cdots + v_n e_n \mapsto (v_1 \cdots v_n)$$

$$|v|^2 = \sum_{i=1}^{n} |v_i|^2$$

Baby example of identification of V with V^*

Let V be a vector space over $\mathbb{R}$

e.g. $V = \mathbb{R}^2$, equip $\mathbb{R}^2$ with std. $\langle -,- \rangle$ Euclidean product

- For each $v \in V$, we can define an element $\phi_v \in V^*$ as.
  $\phi_v(w) = \langle v, w \rangle$ $\quad w \in V$.
- This gives a map of $\overset{\mathbb{R} \text{ vector space}}{\text{set}}$ $V \xrightarrow{\sim} V^*$, $v \mapsto \phi_v$

  isomorphism

  pf: ① injective map

  $\rightarrow$ sending v to $\ker(V^*)$ × lose info.

Similarly, for V hermitian vector space.

we have a map of sets Φ: $V \to V^*$

$$v \mapsto \phi_v = \langle v, - \rangle$$

Why not $\langle -, v \rangle$?

Suppose $\varphi_v = \langle -, v\rangle$

$\varphi_v : V \to \mathbb{C}$

$w \mapsto \langle w, v\rangle$

this φ_v is not a $\mathbb{C}$-linear map

$\varphi_v(\lambda \cdot w) = \bar{\lambda} \cdot \varphi_v(w)$

Φ is not $\mathbb{C}$-linear. but anti $\mathbb{C}$-linear

$\Phi(v) = \phi_v$

$\Phi(i \cdot v) = -i\phi_v$.

$\Phi(\lambda v) = \bar{\lambda}\phi_v$.

Adjoint map:

Given a linear map of hermitian vector space

$A: V \to W$

$\Rightarrow$ $A^* : W^* \to V^*$ (adjoint map of dual space)

$$\begin{tikzcd} W^* \arrow[r, "A^*"] & V^* \arrow[d, "\Phi_V^{-1}"] \\ W \arrow[u, "\Phi_W", "\simeq"'] \arrow[r, "A^+/A^*"'] & V \end{tikzcd}$$

$\Rightarrow$

two anti-$\mathbb{C}$-linear form $W \to W^*$, $V^* \to V$ cancels out

(Here $A^* = A^\dagger$)

Defining property of A^*: Given $v \in V$, $w \in W$

$\langle w, Av\rangle_W = \langle A^* w, v\rangle_V$

Concretely. if $A: \mathbb{C}^n \to \mathbb{C}^m$

hermitian vector space.

A has matrix A_{ij}

then $A^* : \mathbb{C}^m \to \mathbb{C}^n$.

has matrix $(A^*)_{ij} = \overline{A_{ji}}$

Normal Operator:

Let $(V, \langle -, - \rangle)$ be a Hermitian v.s.

$A: V \to V$ $\mathbb{C}$-linear map.

we say A is normal if A^* commute with A

$\Leftrightarrow A^* \cdot A = A \cdot A^*$

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[rr, bend right, "A^*A"'] & V \arrow[r, "A^*"] & V \end{tikzcd}$$

Ex. · if $A^* = A$, A is self-adjoint

A is normal.

· if $A^* = A^{-1}$, A is unitary operator

A is normal

$A^*A = AA^* = I$

· $A^* = -A$. A is skew-adjoint

normal

Ex. $A: \mathbb{C}^2 \to \mathbb{C}^2$.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$A^*A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$AA^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

× normal

Thm: Spectral thm for Normal Operator

Let $A: V \to V$ linear map of hermitian v.s.

Then A is normal $\Longleftrightarrow$ $\exists$ ONB $\{e_1 \cdots e_n\}$ s.t. $Ae_i = \lambda_i e_i$

Pf: sketch: A is represented by a diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad A^* = \begin{pmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{pmatrix}$$

$$AA^* = \begin{pmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_n|^2 \end{pmatrix}$$

① Find 1 eigenvalue λ (a sol to $\det(A - \lambda I) = 0$)

$\because$ work in $\mathbb{C}$

$\therefore$ polynomial has roots.

② eigenspace. $W_\lambda = \{v \mid Av = \lambda v\}$

$= \ker(A - \lambda I)$

invariant under multiplication by A

③. $W_\lambda^\perp$ is also invariant under A & A^*

Induction:

suppose the problem solved for

$V' = W_\lambda^\perp$, $A' = A|_{W_\lambda^\perp}$

Abstract version:

- V : complex v.s./$\mathbb{C}$ of dim n.
- $\langle -,- \rangle$: pos. def hermitian form on V
- $A: V \to V$ $\mathbb{C}$-linear operator

  $A^*: V \to V$ adjoint

(A^* is defined s.t. $\forall v_1, v_2 \in V$

$\langle v_1, Av_2 \rangle = \langle A^* v_1, v_2 \rangle$)

- A is normal, if $[A, A^*] = 0$

  $A, B: V \to V$

  $[A \circ B - B \circ A] := [A, B]$.

Thm (Spectral Thm for normal operators)

A is normal $\Longleftrightarrow$ $\exists$ a ONB of V consist of eigenvectors for A.

Concrete version:

- $V = \mathbb{C}^n$ $\quad \langle -,- \rangle : V \times V \to \mathbb{C}$

  $\langle (z_1 \cdots z_n), (w_1, \cdots w_n) \rangle = \bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_n w_n$

- A: $n \times n$ matrix with entries in $\mathbb{C}$

  given $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}^n$, apply A to z to get Az.

- A^* is the hermitian conjugate of A
- A is normal if $AA^* = A^*A$

e.g. $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ $\quad A^* = \bar{A}^t = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{21} \\ \bar{A}_{12} & \bar{A}_{22} \end{bmatrix}$

$[\bar{A}_{ij}^t] = [\bar{A}_{ji}]$

Thm(concrete version)

If A is an $n \times n$ matrix s.t. $[A, A^*] = 0$, then there exist a unitary matrix U s.t. $U^* A U$ is diagonal

Connection:

From abstract to concrete:

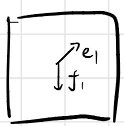
- Find an arbitrary ONB for V, $\{e_1 \cdots e_n\}$
  (with nothing to do with A)
  (e_i) allows one to identify $\Phi: V \to \mathbb{C}^n$.
  $\langle v_1, v_2 \rangle_V = \langle \Phi(v_1), \Phi(v_2) \rangle_{\mathbb{C}^n}$ preserving $\langle -,- \rangle$

- Let $\{f_1 \cdots f_n\}$ be another ONB for V, s.t. $A(f_i) = \lambda_i f_i$ $\lambda_i \in \mathbb{C}$
  then $(e_1 \cdots e_n) = (f_1 \cdots f_n) \underbrace{\begin{pmatrix} u_{11} & u_{12} & \cdots \\ u_{21} & & \\ \vdots & & \\ \vdots & & u_{nn} \end{pmatrix}}_{u}$

  u is also unitary
  $u^* = u^{-1}$

Lemma: if $(e_i)_{i=1}^n$, $(f_i)_{i=1}^n$ are ONB for V
then $(e_1 \cdots e_n) = (f_1 \cdots f_n)\, u$ with u a unitary matrix

Ex: $V = \mathbb{C}$



$\mathbb{C}$

e_1 ONB $\Leftrightarrow$ $|e_1| = 1$
f_1 ONB $\Leftrightarrow$ $|f_1| = 1$
$e_1 = f_1 \cdot e^{i\theta}$

$u = e^{i\theta}$ $u^* = e^{-i\theta}$.
$u \cdot u^* = 1$

Suppose $e_1 = e^{i\theta_1}$, $f_1 = e^{i\theta_2}$
then $\langle e_1, f_1 \rangle = e^{-i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_2 - \theta_1)}$
$\langle f_1, e_1 \rangle = e^{i(\theta_1 - \theta_2)}$
$e_1 = f_1 \cdot e^{i\theta} \Leftrightarrow e^{i\theta_1} = e^{i\theta_2} \cdot e^{i(\theta_1 - \theta_2)}$
$e^{i\theta} = e^{i(\theta_1 - \theta_2)} = \langle f_1, e_1 \rangle$
$e_1 = \langle f_1, e_1 \rangle \cdot f_1$ $v = \langle f_1, v \rangle f_1$

If (e_i) is an ONB of V then $\forall v \in V$
$v = \langle e_1, v \rangle e_1 + \cdots \langle e_n, v \rangle e_n$
$e_i = \sum_{j=1}^{n} \langle f_j, e_i \rangle f_j$

$(e_1 \cdots e_n) = (f_1 \cdots f_n) \underbrace{\begin{pmatrix} \langle f_1, e_1 \rangle & \cdots & \cdot \\ \langle f_2, e_1 \rangle & & \\ \vdots & & \\ \langle f_n, e_1 \rangle & & \langle f_n, e_n \rangle \end{pmatrix}}_{u}$

$u_{ij} = \langle f_i, e_j \rangle$. $(u^*)_{ij} = \overline{u_{ji}} = \overline{\langle f_j, e_i \rangle}$
$(u \cdot u^*)_{ij} = \sum_{k=1}^{n} u_{ik} (u^*)_{kj}$
$= \sum_{k=1}^{n} \langle f_i, e_k \rangle \langle e_k, f_j \rangle = \langle f_i, f_j \rangle$

$[\ \langle w, v \rangle = \langle w, \sum_k \langle e_k, v \rangle e_k \rangle = \sum_k \langle e_k, v \rangle \langle w, e_k \rangle$
$= \sum_k \langle w, e_k \rangle \langle e_k, v \rangle$

Proof of Spectral thm (abstract)

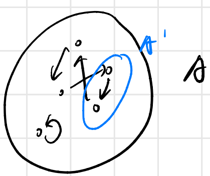
A is normal $\iff$ $\exists$ a ONB of V consist of eigenvectors in A

(

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ A' \arrow[u, phantom, "\cup"] \arrow[ur] & \end{tikzcd}$$

operator : $A \circlearrowleft$ map to itself.

$$\begin{tikzcd} A \arrow[r, "f"] & A \\ A' \arrow[u, phantom, "\cup"] \arrow[ur, "f"] \arrow[r, dashed] & A' \end{tikzcd}$$



)

step 1: ① want to find an eigenvector (actually eigenspace)

if $A \cdot v = \lambda \cdot v$

then $(A - \lambda I) \cdot v = 0$.

i.e. $v \in \ker(A - \lambda I)$ (rank-nullity)

$\Rightarrow \det(A - \lambda I) = 0$.

Solve $\det(A - \lambda I) = 0$

$\quad\quad \| $ degree n polynomial in λ

$\because$ working in $\mathbb{C}$

$\therefore$ By Fundamental Thm of Algebra

$\exists$ factorization $\det(A - \lambda I)$

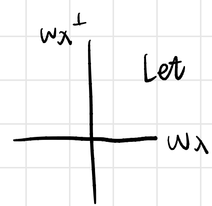
$= c(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$

pick one sol'n to $\det(A - \lambda I) = 0$

say $\lambda = \lambda_1$

then let $W_\lambda = \ker(A - \lambda I)$

$= \{ v \in V \mid (A - \lambda I) v = 0 \}$



Let $W_\lambda^\perp = \{ w \in V \mid \langle w, v \rangle = 0, \forall v \in W \}$

② $A = A^*$ presence W_λ

- $\forall v \in W_\lambda$, we need to show $A \cdot v \in W_\lambda$, $A^* v \in W_\lambda$

$$Av \in W_\lambda \iff A(Av) = \lambda \cdot (Av)$$
$$\iff A \cdot (\lambda v) = \lambda \cdot Av \quad \checkmark$$

$$A^* v \in W_\lambda \iff A \cdot (A^* v) = \lambda \cdot (A^* v)$$
$$\iff A^* A v = \lambda A^* v$$
$$\iff A^* (\lambda v) = \lambda A^* v \quad \checkmark$$

③. A and A^* preserves $W_\lambda^\perp$

Suppose $w \in W_\lambda^\perp$, then $Aw \in W_\lambda^\perp$, and $A^* w \in W_\lambda^\perp$

$$Aw \in W_\lambda^\perp \iff \forall v \in W_\lambda, \quad \langle v, Aw \rangle = 0$$
$$\iff \forall v \in W_\lambda, \quad \langle A^* v, w \rangle = 0.$$

this time, since $A^* v \in W_\lambda$, $w \in W_\lambda^\perp$

$$A^* w \in W_\lambda^\perp \iff \forall v \in W_\lambda, \ \langle v, A^* w \rangle = 0$$
$$\iff \forall v \in W_\lambda, \ \langle Av, w \rangle = 0$$

$\because$ $Av \in W_\lambda$, $w \in W_\lambda^\perp$

$\therefore$ this holds.

From $\mathbb{R}$-v.s. to $\mathbb{C}$ v.s. Complexification

$\mathbb{R} \rightsquigarrow \mathbb{C}$

$\mathbb{R}^n \rightsquigarrow \mathbb{C}^n$

Let $V_{\mathbb{R}}$ be a $\mathbb{R}$ v.s. of $\dim_{\mathbb{R}} n$

we define $V_{\mathbb{C}} \left\{ x+iy \;\middle|\; \begin{matrix} x \in V_{\mathbb{R}} \\ y \in V_{\mathbb{R}} \end{matrix} \right\}$

as a set $\simeq$ $V_{\mathbb{R}} \times V_{\mathbb{R}}$ (x, y)

let $z = a+ib$, $a, b \in \mathbb{R}$

then $v = x+iy \in V_{\mathbb{C}}$

$$z \cdot v = (a+ib)(x+iy)$$
$$= (ax-by) + i(bx+ay)$$

$\mathbb{R}$-vector space

inner product space
$\begin{cases} \bullet\ V \simeq \mathbb{R}^n \\ \bullet\ \langle\ ,\ \rangle : \text{positive definite symmetric bilinear form on } V \end{cases}$

$\langle v, v\rangle > 0, \ \forall v \neq 0$

- $A: V \to W$

  $A^*: W \to V$ adjoint map

$$\begin{tikzcd} W^* \arrow[r, "A^*"] & V^* \arrow[d, "\simeq"] \\ W \arrow[u, "\simeq"] \arrow[r, dashed] & V \end{tikzcd}$$

[concretely, given ONB of V, W,

$A^* = A^t$]

Complexify inner product

- $V_{\mathbb{C}}$
- $\langle -, - \rangle_{V_{\mathbb{C}}}$, having ONB of $V_{\mathbb{R}}$ as ONB of $V_{\mathbb{C}}$

$\mathbb{R} \hookrightarrow \mathbb{C}$

$V_{\mathbb{R}} \hookrightarrow V_{\mathbb{C}}$

$x \longmapsto (x + i0)$

$$\begin{tikzcd} \mathbb{R}^n \arrow[d, hook] & V_{\mathbb{R}} \arrow[l, "\simeq"'] \arrow[d, hook] \\ \mathbb{C}^n & V_{\mathbb{C}} \arrow[l] \end{tikzcd}$$

- In one word: complexification of $V_{\mathbb{R}}$ with basis $(e_1 \cdots e_n)$ is to allow $\mathbb{C}$-coeff $c_1e_1 + \cdots c_ne_n$, $c_i \in \mathbb{C}$

Thm: Let $(V_{\mathbb{R}}, \langle -,- \rangle)$ be a real vector space
Let $(V_{\mathbb{C}}, \langle -,- \rangle)$ be its complexification
suppose $A_{\mathbb{R}}: V_{\mathbb{R}} \to V_{\mathbb{R}}$ is a normal operator
$[A_{\mathbb{R}}, A_{\mathbb{R}}^*] = 0$
Then, its complexification $A_{\mathbb{C}}: V_{\mathbb{C}} \to V_{\mathbb{C}}$ has eigenvectors in $V_{\mathbb{C}}$

need to start from $\mathbb{R}$ to take complex conjugate.

i.e. $\exists$ eigenvectors of A $\{e_1 \cdots e_n\} \subset V_{\mathbb{C}}$ that forms a ONB of $V_{\mathbb{C}}$

- if eigenvalue $\lambda_i \notin \mathbb{R}$, then $\bar{\lambda}_i$ is also an eigenvalue

and if e_i is e.v. $A \cdot e_i = \lambda_i \cdot e_i$
then $\bar{e}_i$ is e.v. $A \bar{e}_i = \bar{\lambda}_i \cdot \bar{e}_i$

e.g. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = A$

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = 0$$

$\Leftrightarrow \lambda^2 + 1 = 0 \quad \lambda = \pm i$

$\lambda = i \quad \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} v = 0$

$v = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \xRightarrow{\text{take conjugate.}} \quad \bar{\lambda} = -i$

$\bar{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Bilinear form

- Def
  - $f: V \times V \to R$
  - $B(x, y) = x^t B y = \sum_{j=1}^{m} \sum_{i=1}^{n} x_j b_{ji} y_i$
- prop
  - $b_{ij} = (e_i, e_j)$
  - change of basis from std to $v_1 \cdots v_n$
    - $B^t A B$, $B = ((v_1)(v_2) \cdots (v_n))$
- Over R^n
  - linear in both slots
  - symmetric $\Leftrightarrow A = A^t$
  - $B(x, y) = x^t B y$
- over C^n (sesquilinear)
  - anti-linear in 1st, linear in 2nd.
  - Hermitian $\Leftrightarrow A = A^*$
  - $S(x, y) = \overline{x}^t B y = x^* B y$

Positive Definite — Def

- $B(v, v) > 0, \forall v$
- $\Updownarrow$
- P.D. matrix: $x^t A x > 0, \forall x \neq 0$

Differential Equation :

want to solve unknown function: $f(x)$ / $x(t)$

- ODE : function with one varible.
- PDE : else.

$$\frac{dx}{dt}$$
$$\uparrow$$

linear ODE : equation involving only $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$ ...

not $x^2(t)$ $x^n(t)$

non-linear ODE : ex : $\frac{dx(t)}{dt} = x(t)^2$

Ex : (1) $\frac{dx}{dt} = 0$

soln : $x(t) = \text{const}$

(2) $\frac{dx}{dt} = C_1$

soln : $x(t) = C_0 + C_1 t$

(3) $\frac{dx}{dt} = f(t)$

$$x(t) = \int^t f(t')dt' + C$$
$$= \int_{t_0}^t f(s)ds + x(t_0)$$

(4) $\dot{x}(t) = x(t)$

soln : $x(t) = C_0 e^t$
$= x(t_0) \cdot e^{t-t_0}$

$$\frac{dx}{dt} = x \iff \frac{dx}{x} = dt \iff d(\ln x) = dt \iff \ln x = t + C \iff x = e^{t+C_0}$$

$$\uparrow$$

$$\frac{d\ln x}{dx} = \frac{1}{x}$$

(or $d(\ln x - t) = 0$.
$\ln x - t = C$)

$$= \underbrace{e^{C_0}}_{C} \cdot e^t$$

(5) $\dot{x}(t) = x^2(t)$

Soln: $\frac{dx}{dt} = x^2$

$\Leftrightarrow \frac{dx}{x^2} = dt$ $\qquad \frac{d(-\frac{1}{x})}{dx} = \frac{1}{x^2}$

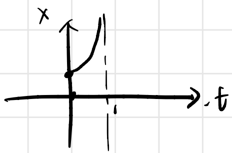
$\Leftrightarrow d(-\frac{1}{x}) = dt$

$\Leftrightarrow -\frac{1}{x} = t + c$

$\Leftrightarrow x = \frac{-1}{t+c}$

$= \frac{1}{(-c)-t}$

suppose $t=0$, $x=1$ $\Rightarrow c=-1 \Rightarrow x = \frac{1}{1-t}$



trick: separation of variables:

x and t on two sides.

(6) ☆ $\frac{dx}{dt} = x + t$

try with $x(t) = u(t)e^t$.

new unknown function

$\frac{d(u(t)e^t)}{dt} = u(t)\cdot e^t + t$

$u = u(t)$ $\quad \dot{u}e^t + u\cdot e^t = u\cdot e^t + t$

$\dot{u}e^t = t$

$\dot{u} = t\cdot e^{-t}$

$u(t) = u(t_0) + \int_{t_0}^{t} s\cdot e^{-s}\cdot ds$

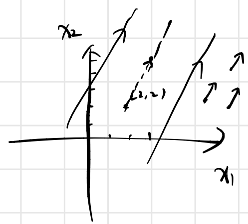
$= C + \int te^{-t}dt$

unknown functions: $x_1(t)$, $x_2(t)$

equations: $\begin{cases} \dot{x}_1 = F_1(x_1, x_2, t) \\ \dot{x}_2 = F_2(x_1, x_2, t) \end{cases}$

still ODE cuz $\dot{x}_1 = \frac{x_1(t)}{dt}$ $\dot{x}_2 = \frac{x_2(t)}{dt}$.

Ex. (1). $\begin{cases} \dot{x}_1 = 1 \\ \dot{x}_2 = 3 \end{cases}$



vector field

functions input position output vector

if we have ~~boundary~~ condition initial $\begin{cases} x_1 = 2 \\ x_2 = 2 \end{cases}$.

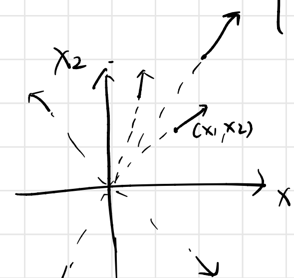
$x_1 = 2 + 1 \cdot t$

$x_2 = 2 + 3 \cdot t$

(2). $\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_2 \end{cases}$ soln $\begin{cases} x_1 = C_1 e^t \\ x_2 = C_2 \cdot e^t \end{cases}$

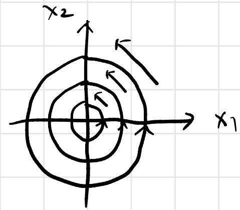
with initial condition $t_0 = 0$

$\begin{cases} x_1 = x_1(t_0) \cdot e^t \\ x_2 = x_2(t_0) \cdot e^t \end{cases}$



further away $\Rightarrow$ larger vector

(3)



integral curve

given soln

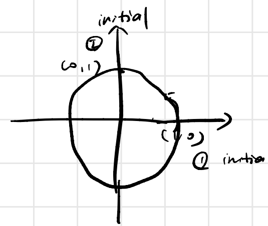
↓

equation $\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \end{cases}$

one soln ① $\begin{cases} x_1 = \cos(t) \\ x_2 = \sin(t) \end{cases}$ $\dot{x}_1 = -\sin t = -x_2$, $\dot{x}_2 = \cos(t) = x_1$

another ② $\begin{cases} x_1 = -\sin(t) \\ x_2 = \cos(t) \end{cases}$

if $t=0$



$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} \swarrow \text{sol 1}$$

$$= \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \swarrow \text{sol 2.}$$

Claim: general soln = $c_1 \cdot sol_1 + c_2 \cdot sol_2$

Eq $\iff \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\iff \left[\frac{d}{dt} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0$

operator on function

$\iff \begin{pmatrix} \frac{d}{dt} & 1 \\ -1 & \frac{d}{dt} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0$

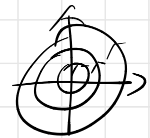
solutions are in the kernel of operator

addition of solution are still in kernel.

General ex:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightsquigarrow$$ sol'n : radial motion outward

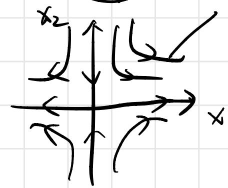
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightsquigarrow$$



odd ball $\Rightarrow$ spectral thm

conjugate eigenvalues

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightsquigarrow$$



$x_1 \cdot x_2 = \text{const}$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} x_1 = e^{\lambda_1 t} c_1 \\ x_2 = e^{\lambda_2 t} c_2 \end{cases}$$

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

$\frac{d}{dt}x(t) = ax+b$. $\leftarrow$ linear ODE

$\uparrow$ only linear term & const term appear, no x^2

$\uparrow$ t

Special Ex: $\frac{d}{dt}x(t) = x(t) \rightsquigarrow x(t) = c\cdot e^t$

$$\frac{dx}{dt} = \lambda x \xrightarrow{\lambda\in\mathbb{C}} x(t) = c\cdot e^{\lambda t}$$

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t)\end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ & & & \lambda_n\end{pmatrix}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t)\end{pmatrix}$$

$$\Longleftrightarrow \begin{cases} \frac{d}{dt}x_1(t) = \lambda_1 x_1(t) \\ \vdots \\ \frac{d}{dt}x_n(t) = \lambda_n x_n(t)\end{cases}$$

$$\Rightarrow x_k(t) = c_k e^{\lambda_k t}$$

$$\frac{d}{dt}\vec{x} = \underset{\text{diagonal matrix lambda.}}{\underset{\downarrow}{\Lambda}} x$$

$$\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t)\end{pmatrix} = \begin{pmatrix} & C & \end{pmatrix}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t)\end{pmatrix}$$

$\uparrow$ invertible matrix constant

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ \vdots \\ y_n\end{pmatrix} = \frac{d}{dt}(C\cdot\vec{x}(t)) = C\cdot\frac{d}{dt}\vec{x}(t)$$

$$= C\cdot\Lambda x(t)$$

$$= (C\cdot\Lambda C^{-1})y(t)$$

Solving steps: start with $\frac{d}{dt}y = A\cdot y$

① try to diagonalize A as

$$A = C\Lambda C^{-1}$$

② Then we have

$$\begin{cases} x(t) = C^{-1}y \\ \frac{d}{dt}x = \Lambda x\end{cases}$$

$$\Rightarrow x = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t}\end{pmatrix}$$

$$y = \begin{pmatrix} C \end{pmatrix}\begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t}\end{pmatrix}$$

Ex: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\det(A - \lambda I) = 0$ characteristic polynomial

$\Leftrightarrow \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = (-\lambda)^2 + 1 = 0$

$\lambda = \pm i$

for $\lambda = i$ $\quad \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

one soln $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} \leftarrow v_1$

$(1 \ \ -i) \begin{pmatrix} i \\ 1 \end{pmatrix} = 0$

for $\lambda = -i$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} \leftarrow v_2$

$A v_1 = \lambda_1 v_1$
$A v_2 = \lambda_2 v_2$

$$A \left((v_1)(v_2) \right) = \left((v_1)(v_2) \right) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

let $C = \left((v_1), (v_2) \right)$

$$AC = C\Lambda$$
$$A = C \Lambda C^{-1}$$

$$\underset{A}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \underset{C}{\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}} \underset{\Lambda}{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}} \underset{C^{-1}}{\left(\frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \right)}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

set $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

then $\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} i & \\ & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Rightarrow \quad x_1 = C_1 \cdot e^{it}$
$\qquad x_2 = C_2 \cdot e^{-it}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \cdot e^{it} \\ C_2 \cdot e^{-it} \end{pmatrix}$$

What if $A \neq C\Lambda C^{-1}$

can't be diagonalized?

e.g. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = 0 \end{cases}$$

$$\begin{cases} x_2(t) = C_2 \\ x_1(t) = C_2 t + C_1 \end{cases}$$

Techique: one can always reduce an equation about higher order derivates $\left(\frac{d}{dt}\right)^k$ to first order derivates, at the cost of introducing more variables

Ex. $F = ma$

$C = m \cdot \left(\frac{d}{dt}\right)^2 x(t)$

$\ddot{x}(t) = C \iff \begin{array}{l} x_1(t) = x(t) \\ x_2(t) = \dot{x}(t) \end{array}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = C \end{cases}$$

Standard A : translate other into A

- $A = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$ $n \times n$

$$\frac{d}{dt} x_{(t)} = A\, x(t)$$

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = 0 \end{cases} \Leftrightarrow \left(\frac{d}{dt}\right)^n x_1(t) = 0$$

$x_1(t) = t^{n-1} C_{n-1} + \cdots + t \cdot C_1 + C_0$

$x_2(t) = \dot{x}_1 = \cdots$

Try to diagonalize matrix into either into $\frac{d}{dt} x = \lambda x$ or .

Ex. · fmixed up

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Bigg(\lambda \cdot Id + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{N}\Bigg) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Nilpotent

$$\frac{d}{dt} \vec{x} = (\lambda + N) \vec{x}$$

Recall:

easier case: $\frac{d}{dt} x = \lambda x \Rightarrow x = C \cdot e^{\lambda t}$

harder case: $\frac{d}{dt} x = \begin{pmatrix} 0 & 1 & \\ & \ddots & 1 \\ & & 0 \end{pmatrix} x \Rightarrow x_1(t) = t^{n-1} C_{n-1} \cdots + t C_1 + C_0$

Goal: remove λ on the diagonal

Soln: set $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$

$$\vec{x} = e^{\lambda t} \vec{u}(t)$$

$$\frac{d}{dt}(e^{\lambda t} \cdot u) = (\lambda + N) \cdot (e^{\lambda t} \cdot u)$$

$$\lambda \cdot e^{\lambda t} u + e^{\lambda t} \dot{u} = \lambda e^{\lambda t} u + N \cdot e^{\lambda t} \cdot u.$$

$$e^{\lambda t} \dot{u} = e^{\lambda t} \cdot N \cdot u$$

$$\Leftrightarrow \quad \dot{u} = N u.$$

Solve $u \rightarrow$ substitute back to x

In general, if $N_n = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$

then $\frac{d}{dt}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (\lambda + N_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ can be solved.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = e^{\lambda t}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$\Leftrightarrow \left(\frac{d}{dt}\right)^n u_1(t) = 0.$

$\Leftrightarrow u_1(t) =$ polynomial of degree $n-1$

Jordan Decomposition Thm. $/\mathbb{C}$

- A: $n\times n$ matrix

$\Rightarrow \exists$ invertibe matrix C s.t. $A = C \left(\underbrace{\begin{matrix} \boxed{\lambda_1+N_1} & & \\ & \boxed{\lambda_2+N_2} & \\ & & \boxed{\lambda_r+N_r} \end{matrix}}_{r \text{ blocks}.\ \lambda_i\neq\lambda_j} \right) C^{-1}$

$N_1 = (0)$ $N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $N_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Pf: Step 1: how to get $(\lambda_1 \cdots \lambda_r)$

Solve the equation $\det(A-\lambda I) = 0$

get eigenvalue decomposition

$\det(A-\lambda I) = C\cdot(\lambda-a_1)^{m_1}\cdots(\lambda-a_k)^{m_k}$

For each root a_i, we can define generalized eigenspace.

$W_{a_i} = \ker\left((A-a_i)^{m_i}\right)$ (*Normal: $\{v : (A-a)v = 0\}$

Ex. $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} = 3\cdot Id + N_3$

$\det(A-\lambda I) = \begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (3-\lambda)^3$

$A-3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$N_3^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$N_3^3 = 0$

$\ker(N_3) = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{C}^3 \,\middle|\, N_3\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \right\}$

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_2 \\ v_3 \\ 0 \end{pmatrix} = 0$

$= \mathrm{span}(e_1)$

$\ker(N_3^2) = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \,\middle|\, v_3 = 0 \right\}$

$= \mathrm{span}(e_1, e_2)$

$\ker(N_3^3) = \mathbb{C}^3$

Ex 2: $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \boxed{3+N_1} & \\ & \boxed{3} \end{pmatrix}$

$$\det(A - \lambda I) = \begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (3-\lambda)^3$$

$$A - 3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{0} \end{pmatrix} \Rightarrow (A-3)^2 = 0.$$

$\ker(A-3) = \mathrm{span}(e_1, e_3)$

$\ker((A-3)^2) = \mathbb{C}^3$

Conclusion: If A: $n \times n$ matrix / $\mathbb{C}$

$$\det(A - \lambda I) = c(\lambda - a_1)^{m_1} \cdots (\lambda - a_n)^{m_n}$$

$$W_{a_i} = \ker((A - a_i)^{m_i})$$

- Now $A|_{W_{a_i}}$ has only eigenvalue a_i

$(A - a_i)|_{W_{a_i}}$ is nilpotent, i.e.

Sufficient high power $\left((A - a_i)|_{W_{a_i}}\right)^N = 0$.

Ex. of nilpotent matrix of size n.

$$N_4 = \boxed{\begin{matrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{matrix}}$$

← regular nilpotent if $N^1 \neq 0, N^2 \neq 0, N^{n-1} \neq 0$, $N^n = 0$.

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 0 & \\ & & & & 1 \\ & & & & 0 \end{pmatrix}$$

← × regular nilp.

N nilpotent $\Leftrightarrow$ CNC^{-1} nlp.

similar transformation

because $N^k = 0 \Leftrightarrow (CNC^{-1})^k = 0$

Jordan Normal Form / k.

Let k be any field ($\mathbb{Q}$, $\mathbb{C}$, or $\mathbb{F}_q$)
Let V be finite dim v.s. over k.
Let $T: V \to V \in L(V)$
we want to classify T up to "similarity transformation"

T and $\tilde{T}$ similar if $\exists\ V \xrightarrow{C} V$ invertible transformation
s.t. $T = C \cdot \tilde{T} \cdot C^{-1}$

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ C \downarrow \cong & & \cong \downarrow C \\ V & \xrightarrow{\tilde{T}} & V \end{array}$$

Equivalent to:

find nice basis of V s.t. T "look as diagonal as possible"

invariant:

- Characteristic polynomial on $T: V \to V$

$$\det(\lambda I - T) = \lambda^n + p_1 \cdot \lambda^{n-1} + \cdots + p_n$$
$$n := \dim_k V$$

[determinant for a linear operator $A: V \to V$

- pick any basis $e_1 \cdots e_n$ of V.
then A become a matrix s.t.
$Ae_i = A_{i1} e_1 + \cdots A_{in} e_n$

- $\det(A) = \det([A])$

- if choose a different basis
say $\tilde{e}_1 \cdots \tilde{e}_n$ then corresponding matrix
$[\tilde{A}] = C[A]C^{-1}$

thus $\det([\tilde{A}]) = \det(C \cdot [A] C^{-1})$
$= \det(C) \cdot \det(C)^{-1} \cdot \det[A]$
$= \det[A]$

- Assume $\lambda_0 \in k$ is a root of the characteristic polynomial
  i.e. $\det(\lambda_0 I - T) = 0$

  - If $x \in V$ satisfies $Tx = \lambda_0 \cdot x$ and $x \neq 0$
    then x is an eigenvector of T with eigenvalue λ_0

  - $\mathrm{kernal}(\lambda_0 I - T)$: eigenspace of T with eigenvalue λ_0.
    $:= \{x \in V \mid Tx = \lambda_0 x\}$

  - root space (generalized eigenspace) of eigenvalue λ_0

Consider $T - \lambda_0 I$: operator $V \to V$
and $(T-\lambda_0 I)^2 \cdots$

if $(T-\lambda_0 I)^k x = 0$. then $(T-\lambda_0 I)^{k+1} x = (T-\lambda_0 I)\underbrace{(T-\lambda_0 I)^k x}_{0}$
$= 0$.

$\therefore \ker(T-\lambda_0 I) \cdots \subset \ker((T-\lambda_0 I)^k) \subset \ker((T-\lambda_0 I)^{k+1}) \subset \cdots$

Since they are all subspaces of V finite dim.
we know for certain m,

Define:
$$W_{\lambda_0} := \ker((T-\lambda)^m) = \ker((T-\lambda)^{n+1}) = \cdots$$
root space for λ_0

$$\mathcal{U}_{\lambda_0} := \mathrm{im}((T-\lambda_0)^n)$$

Lemma: • W_{λ_0} and U_{λ_0} are T-invariant

• $W_{\lambda_0} \cap U_{\lambda_0} = \{0\}$

• $V = W_{\lambda_0} \oplus U_{\lambda_0}$

Pf: ①. If $v \in W_{\lambda_0}$, then $(T-\lambda_0)^m v = 0$.

$(T-\lambda_0 I)^m (Tv) = T \cdot (T-\lambda_0 I)^m v = 0$.

If $v \in U_{\lambda_0}$ then $\exists \tilde{v} \in V$, s.t.

$v = (T-\lambda_0 I)^m \cdot \tilde{v}$

$Tv = (T-\lambda_0 I)^m \cdot (T\tilde{v}) \in U_{\lambda_0}$

②. Suppose $v \neq 0 \in U_{\lambda_0} \cap W_{\lambda_0}$ then

$$\begin{cases} v = (T-\lambda_0 I)^m \cdot \tilde{v} \\ (T-\lambda_0 I)^m v = 0 \end{cases}$$

$\Rightarrow$ $(T-\lambda_0 I)^{2m} \tilde{v} = 0$

$\tilde{v} \in \ker((T-\lambda_0 I)^{2m})$

but $\tilde{v} \notin \ker((T-\lambda_0 I)^m)$

contradict with $\ker((T-\lambda_0 I)^m) = \ker((T-\lambda_0 I)^{m+1}) \cdots$

③. Recall for any linear map $A: V_1 \to V_2$

$\dim V_1 = \dim \ker A + \dim \operatorname{im} A$

apply this to $(T-\lambda_0 I)^m: V \to V$

• $\dim V = \dim W_{\lambda_0} + \dim U_{\lambda_0}$

• Recall for any 2 vector subspace $V_1, V_2 \subset V$

$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

thus $\dim(W_{\lambda_0} + U_{\lambda_0}) = \dim W_{\lambda_0} + \dim U_{\lambda_0}$

$= \dim V$

$V = W_{\lambda_0} + U_{\lambda_0}$.

$= W_{\lambda_0} \oplus U_{\lambda_0}$.

Assume $\det(\lambda I - T) = (\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_r)^{m_r}$

(True when over $\mathbb{C}$)

Want to prove

lemma : $V = W_{\lambda_1} \oplus W_{\lambda_2} \oplus \cdots \oplus W_{\lambda_r}$

Pf: $V = W_{\lambda_1} \oplus U_{\lambda_1}$ by previous lemma.

say $W_{\lambda_1} := \ker((T-\lambda_1)^{N_1})$ $\quad U_{\lambda_1} = \operatorname{im}((T-\lambda_1)^{N_1})$

Claim: $W_{\lambda_2} \cdots W_{\lambda_r} \subset U_{\lambda_1}$

If we consider

$T - \lambda_1 I$ restricted to W_{λ_2}

W_{λ_2} is both T-invariant & I invariant

$\therefore$ it preserves W_{λ_2} And $\forall v \in W_{\lambda_2}$, $(\neq 0)$ $(T-\lambda_1)v \neq 0$.

(If $Tv = \lambda_1 v$, $(T-\lambda_2)^{N_2} v$
$= (\lambda_1-\lambda_2)^{N_2} \cdot v \neq 0$)

Thus $T-\lambda_1 |_{W_{\lambda_2}}$ is invertible.

$\therefore (T-\lambda_1)^{N_1}\Big|_{W_{\lambda_2}}$ is invertible.

$W_{\lambda_2} = (T-\lambda_1)^{N_1}(W_{\lambda_2}) \subset (T-\lambda_1)^{N_1} V = U_{\lambda_1}$

This claims shows $W_{\lambda_i} \cap W_{\lambda_j} = \{0\}$

Hence $V = W_{\lambda_1} \oplus \cdots \oplus W_{\lambda_r} \oplus U$.

$U = U_{\lambda_1} \cap \cdots \cap U_{\lambda_r}$, Hence is preserved by T

Claim: If $V = V_1 \oplus V_2$ and $T: V \to V$ preserves V_1 and V_2

then

$\operatorname{char}(T) = \operatorname{char}(T|_{V_1}) \cdot \operatorname{char}(T|_{V_2})$

Pf: Choose a basis of V adapted to $V = V_1 \oplus V_2$ then T matrix is block diagonal

$$T = \begin{array}{cc} & \begin{array}{cc} V_1 & V_2 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \end{array} & \left[\begin{array}{c|c} T_1 & 0 \\ \hline 0 & T_2 \end{array}\right] \end{array}$$

$\det(\lambda - T) = \det\left(\begin{array}{c|c} \lambda - T_1 & 0 \\ \hline 0 & \lambda - T_2 \end{array}\right)$

$= \det(\lambda - T_1) \cdot \det(\lambda - T_2)$

$$\therefore \underset{\substack{\| \\ (\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_r)^{m_r}}}{\mathrm{char}(T)} = \underset{\substack{(\lambda-\lambda_1)^{n_1} \\ \|}}{\mathrm{char}(T|_{W_{\lambda_1}})} \cdot \underset{\substack{(\lambda-\lambda_2)^{n_2} \\ \|}}{\mathrm{char}(T|_{W_{\lambda_2}})} \cdots \mathrm{char}(T|_{W_{\lambda_r}}) \cdot \underset{\substack{\| \\ 1}}{\mathrm{char}(T|_U)}$$

$\mathrm{char}(T|_U) \to$ no roots in $\lambda_1 \cdots \lambda_r$

Thus $U = \{0\}$ $\quad n_i = m_i$

- $T - \lambda_i I$ on W_{λ_i} is a nilpotent operator
  i.e. $(T-\lambda_i I)^N = 0$ on W_{λ_i} for N large enough

- Ex: $N : K^n \to K^n$, $e_1 \cdots e_n$ standard basis.
  $N(e_n) = e_{n-1}$ $\quad N(e_k) = e_{k-1}$, $\cdots$ $N(e_1) = 0$.

$$N = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \ddots & 1 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & 0 \end{pmatrix}$$

such N satisfies $N^n = 0$ but $N^{n-1} \neq 0$.

we call such operator on K^n a regular nilpotent operator.

$N : e_n \mapsto e_{n-1} \mapsto e_{n-2} \cdots e_1 \mapsto 0$

prop: Let V be an n-dim v.s. $/K$
Let $N : V \to V$ be a nilpotent op

then $\exists$ a decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

s.t. $N|_{V_k}$ is a regular nilpotent op.

i.e. $\exists$ a basis of V s.t.

$$N = \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 & \\ & 0 & \ddots \\ & & 0 \end{matrix}} & & 0 \\ & \boxed{\begin{matrix} 0 & 1 & \\ & 0 & \ddots \\ & & 0 \end{matrix}} & \\ 0 & & \ddots \end{bmatrix}$$

of block diagonal form, where each block is a regular nilpotent.

Pf: (method 1) Assume $N^m = 0$, $N^{m-1} \neq 0$.

Then consider $im(N^k)$, if $v \in Im(N^k)$,

then $v = N^k \cdot (\tilde{v})$

$v = N^k(\tilde{v}) = N^j(N^{k-j} v)$ for any $j < k$

Thus $im(N^k) \subset im(N^j)$ for $\forall j < k$

$$V \supset im(N) \supset im(N^2) \cdots \supset im(N^{m-1}) \supset im(N^m) = 0.$$

We are going to construct a basis adapted to this flag in the following sense.

For $im(N^{m-1})$ we find a basis

$im(N^{m-1})$: $e_1, \cdots e_2 \cdots e_k$.

$$\text{m-1 arrows}\left\{\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & & \\ e_1^{(2)} & e_2^{(2)} & \vdots & e_{k+1}^{(2)} \cdots & e_{k+n}^{(2)} \\ \uparrow & \uparrow & \vdots & \uparrow & \\ e_1^{(3)} & e_2^{(3)} & \vdots & e_{k+1}^{(3)} & \\ \vdots & \vdots & \vdots & & \\ e_1^{(m)} & e_2^{(m)} & \vdots & & \end{array}\right.$$

$im(N^{m-2})$ may contain more than $\underbrace{span(e_1 \cdots e_k, e_1^{(1)} \cdots e_k^{(1)})}$

we can complete to a new basis of $im(N^{m-2})$ by adding.

$e_{k+1}^{(2)} \cdots e_{k+n_2}^{(2)}$

Constant coefficient lec.

1. Existence & uniqueness of sol'n

- let $x_1(t), \cdots, x_n(t)$ be unknown function

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

- $$\begin{cases} \dot{x}_1(t) = F_1(t, x_1 \cdots x_n) \\ \vdots \\ \dot{x}_n(t) = F_n(t, x_1 \cdots x_n) \end{cases}$$

  be equations

Thm: suppose $\{F_i(t, \vec{x})\}$ has continuous partial derivatives in t and $x_1 \cdots x_n$. then for any initial condition at $t = t_0$ given by $(x_1(t_0) \cdots x_n(t_0))$, we have a small open integral segment $(t_0 - \varepsilon, t_0 + \varepsilon)$ containing t s.t. the sol'n $\vec{x}(t)$ exists for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ that satisfies eqn & initial condition. Such sol'n is unique.

Ex. a function that doesn't have continuous derivative:

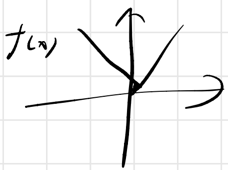
- $f(x) = \frac{1}{x}$, $x \in (-\infty, \infty)$

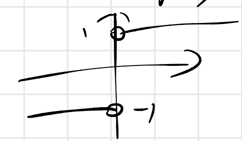
  function not continuous at $x = 0$.

  $\Rightarrow f'(x)$ doesn't exist at $x = 0$.

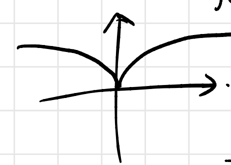
- $f(x) = |x|$

  function is continuous.

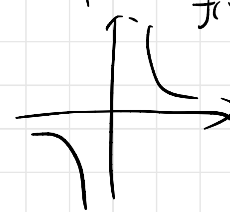




- $f(x) = |x|^{\frac{1}{2}}$



$$f'(x) = \begin{cases} \frac{1}{2} \frac{1}{x} & x > 0 \\ -\frac{1}{2} \frac{1}{x} & x < 0 \end{cases}$$



· Ex: where existence & uniqueness Thm doesn't apply.

$$\begin{cases} \dot{x}(t) = [x(t)]^{\frac{2}{3}} \\ x(0) = 0 \end{cases}$$

we have 2 sol'n

$x(t) = 0$, $\forall t$

or $x(t) = \left(\frac{t}{3}\right)^3$, $\Rightarrow \dot{x}(t) = \left(\frac{t}{3}\right)^2$

Not unique because $F(t, x) = x^{\frac{2}{3}}$ doesn't have continuous derivative at $x = 0$

② $$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \\ - & - & - & - \end{bmatrix}}_{A} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$\{a_{ij}\}$ are constants.

independent of t

homogeneous first-order constant-coefficient ODE

all terms in eq are linear (proportional to) in $\vec{x}$

$\vec{x} \to 2\vec{x}$, all terms change by factor of 2.

$$\frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t)$$

$$\vec{x}(t) = e^{A \cdot t} \cdot \vec{x}(0)$$

Here, $\because A$ is $n \times n$ matrix w/ constant coeff

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

this series converges for $\forall x \in \mathbb{C}$

$$e^{At} = I_n + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

- each term makes sense.

$\because \frac{(At)^n}{n!} = \frac{t^n}{n!} A^n$. well defined.

- summation converges.

$$\frac{d}{dt}(e^{At}) = 0 + A + A^2 t + \cdots A^3 \frac{t^2}{2!}$$

$$= A\left(I + At + A^2\frac{t^2}{2!} \cdots\right)$$

$$= A \cdot e^{At}$$

$$\left[\ \frac{d}{dt} e^{\lambda t} = \lambda \cdot e^{\lambda t}\right.$$

Given $\frac{d}{dt}(e^{At}) = A \cdot e^{At}$

then $\frac{d}{dt}(\underbrace{e^{At} \cdot \vec{x}_0}_{\vec{x}(t)}) = A \cdot \underbrace{e^{At} \cdot \vec{x}_0}_{\vec{x}(t)}$

$\Rightarrow \frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t)$.

Notation: e^{At} is called "fundamental soln"

Since each column vector in e^{At} is a soln

then "general soln" $\vec{x}(t) = e^{At}\vec{x}_0$ is

a linear comb of the columns

In practice, to do exponentiation of a matrix A

we want to diagonalize A first.

Recall: Jordan canonical form Thm

Given a matrix A over $\mathbb{C}$ of size n.

$\exists$ invertible matrix C s.t.

$$A = C \cdot J \cdot C^{-1}$$

where J is block diagonal, with each block

$$\sim \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & \lambda \end{pmatrix}$$

$$A^2 = (CJC^{-1}) \cdot (C \cdot J \cdot C^{-1})$$

$$= C \cdot JJ \cdot C^{-1} = CJ^2C^{-1}$$

$\vdots$

$$A^n = C \cdot J^n \cdot C^{-1}$$

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots$$

$$= C\left[I + Jt + J^2 \cdot \frac{t^2}{2!} \cdots\right] \cdot C^{-1}$$

$$= C \cdot e^{Jt} \cdot C^{-1}$$

Suffice to consider the case where A is of Jordan form

$$\left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right)^2 = \begin{pmatrix} B_1^2 & 0 \\ 0 & B_2^2 \end{pmatrix}$$

Suffice to consider one single Jordan block:

Jordan block size	J	J^n	e^{Jt}
1	λ	λ^n	$e^{\lambda t}$
2	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$		

$$= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \lambda \cdot I_2 + N_2$$

$$(\lambda I + N)^2 = \lambda^2 + 2\lambda N + N^2$$

$$= \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$$

$$(\lambda I + N)^K = \lambda^K + \binom{K}{1}\lambda^{K-1} \cdot N + \cancel{\binom{K}{2}\lambda^{K-2} N^2 + \cdots}$$

$N^n = 0$

$$= \begin{pmatrix} \lambda^K & K\lambda^{K-1} \\ 0 & \lambda^K \end{pmatrix}$$

$\longrightarrow$ $e^{(\lambda+N)t} = I + (\lambda+N)t + (\lambda+N)^2 \frac{t^2}{2} + \cdots$

$$e^{(\lambda t \cdot I) + (t \cdot N)} = e^{\lambda t} \cdot e^{tN}$$

$\Uparrow$ If A, B 2 square matrices & $AB = BA$

then $e^{A+B} = e^A \cdot e^B$

If x, y numbers

$$e^x \cdot e^y = \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + y + \frac{y^2}{2!} + \cdots\right)$$

$$= 1 + (x+y) + \frac{1}{2!}(x^2 + y^2 + 2xy) + \cdots$$

$$= e^{x+y}$$

the only property we use is $xy = yx$.

Hence same arg for matrices

[ODE] 3.6.2

$$e^{tN} = 1 + tN + \frac{t^2}{2}N^2 + \cdots = 1 + tN = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$e^{(\lambda+N)t} = e^{\lambda t} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

So: $J^n = \begin{pmatrix} \lambda^n & n \cdot \lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$ $\quad e^{Jt} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

For $\overset{\text{size } m}{J} = \lambda I_m + N_m = \underbrace{\begin{pmatrix} \lambda & & 1 \\ & \ddots & \\ & & \lambda \end{pmatrix}}_{m}$

$$J^k = \begin{pmatrix} \lambda^k & & 0 \\ & \ddots & \\ 0 & & \lambda^k \end{pmatrix} + k \cdot \lambda^{k-1} \cdot N + \binom{k}{2} \cdot \lambda^{k-2} \cdot N^2 + \cdots$$

$$N_4 = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & 0 & 1 \\ 0 & & & 0 \end{pmatrix} \quad (N_4)^2 = \begin{pmatrix} 0 & 0 & 1 & \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \quad (N_4)^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

$N_4{}^4 = 0$.

$$= \begin{pmatrix} \lambda^k & k \cdot \lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots \\ & \lambda^k & \ddots & \ddots \\ & & \ddots & \ddots \\ & & & \lambda^k \end{pmatrix}$$

Ex: (damped pendulum)

e.g.

$$\ddot{x} = \underset{\substack{\downarrow \\ \text{Hook's law}}}{-kx} + \underbrace{c\cdot\dot{x}}_{\text{resistence}} \qquad k, c > 0$$

Soln: ① turn 2nd order DE to 1st order

Let $x_1(t) = x(t)$

$x_2(t) = \dot{x}_1(t)$

then

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -kx_1 - cx_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{\begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix} t} \cdot \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$\downarrow$ (evolution matrix / propagator)

remember general sol'n :

$$A = \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix}$$

Diagonalize A by $A = C \cdot J \cdot C^{-1}$

Find eigenvalues

$$0 = \det(\lambda I - A) = \begin{pmatrix} \lambda & -1 \\ k & \lambda + c \end{pmatrix} = \lambda(\lambda + c) + k$$

$$= \lambda^2 + c\lambda + k$$

$\ddot{x} + c\cdot\dot{x} + kx = 0.$ try $x = e^{\lambda t}$

solve $\lambda^2 + c\lambda + k = 0$.

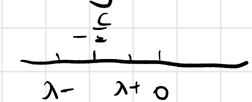
$$\left(\lambda + \frac{c}{2}\right)^2 = \left(\frac{c}{2}\right)^2 - k.$$

$$\lambda = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - k}$$

Case 1: $\lambda_\pm$ are distinct real sol'n

$\Leftrightarrow k < \left(\frac{c}{2}\right)^2$

Intuitively this means friction is large



$$A = C \cdot \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} C^{-1}$$

$$e^{At} = C \cdot \begin{pmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{pmatrix} C^{-1}$$

$$\vec{x}(t) = C \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} C^{-1} \begin{pmatrix} x(0) \\ x_2(0) \end{pmatrix}$$

<u>case 2</u> : $\lambda_\pm$ are a pair of complex conjugate sol'n

$\Leftrightarrow \; k > \left(\frac{c}{2}\right)^2.$

$\lambda_\pm = -\rho \pm i w \qquad w = \sqrt{k - \left(\frac{c}{2}\right)^2}$

$$e^{(-\rho + i w)t} = e^{-\rho t} \left[\cos(wt) + i \sin(wt) \right)$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} C^{-1} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

Last: constant coeff diff E.

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$A \rightarrow$ $n \times n$, constant.

Soln: $\vec{x}(t) = e^{At} \cdot \vec{x}(0)$

$\vec{x}(0)$: a constant columnvector

where exponential of a matrix is understood by its Taylor expansion: $e^{M} = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots$

Today:

1. Boundary condition / initial
2. inhomogeneous term.

1. $\frac{d}{dt}\vec{x} = A\vec{x}$ $\quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Sol'n space is an n-dim vector space

$\Uparrow$

$$\vec{x}(t) = \underbrace{e^{At}}_{n \times n \text{ matrix}} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

c_i: arbitrary constants

we can obtain an isomorphism between $\mathbb{C}^n$ and solution space

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \mapsto e^{At}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

i.e. we can use column vectors in e^{At} as basis of sol'n space

Motivation: need to impose more constraints to pin down sol'n

Ex 1: require that at $t = t_0$.

sol'n $\vec{x}(t_0) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

This can be satisfied by choosing $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

then plug in the general sol'n to the constraints.

$$e^{At_0} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Q: is e^{At_0} invertible m ?

A: Yes.

If A, B commute $AB = BA$

then $e^{A+B} = e^A e^B = e^B \cdot e^A$.

$e^A \cdot e^{-A} = e^0 = I$.

Ex 2:

Eq: $\begin{cases} \ddot{x}(t) = 0 \\ x(0) = 1 \\ x(1) = 2 \end{cases}$

gen sol'n $x(t) = a + bt$ a, b free variables

plug in $\begin{cases} a + b \cdot 0 = 1 \\ a + b \cdot 1 = 2 \end{cases}$

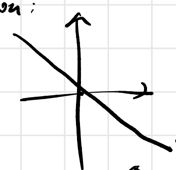
Ex 3: $\begin{cases} \ddot{x}(t) = -x(t) \Rightarrow x(t) = a \cdot \sin t + b \cdot \cos t \\ x(0) = 0 \\ x(2\pi) = 1 \end{cases}$

$\begin{cases} a\sin(0) + b\cos(0) = 0. \\ a\sin(2\pi) + b\cos(2\pi) = 1 \end{cases} \Leftrightarrow \begin{cases} b = 0 \\ b = 1 \end{cases}$

no sol'n

2. inhomogeneous Equation:

Ex: $x+y=0$ $\overset{\text{sol'n}}{\rightsquigarrow}$



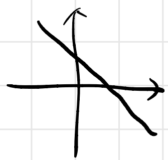
$\leftarrow x+y=0$

$\uparrow$ homogeneous eq.

sol'n is a vector space (subspace across the origin)

prop: Sum of sol'n is still sol'n

$x+y=1$ $\overset{\text{sol'n}}{\rightsquigarrow}$



$\leftarrow$ sol'n not v.s. but affine subspace

Ex: $(1,0)$ and $(0,1)$ are sol'n

but $(1,0)+(0,1)=(1,1)$ not a sol'n

Relation between homogeneous and inhomogeneous:

If (x_0, y_0) is a sol'n to $x+y=1$

and (x_1, y_1) is a sol'n to $x+y=0$

then (x_0+x_1, y_0+y_1) is a sol'n to $x+y=1$

In general:

an inhomogeneous equation is of the form $A\vec{x}=\vec{b}$

$\vec{\cdot}$ $\exists$ a sol'n if $\vec{b}$ is in the range of A.

☆ • If $\vec{x_0}$ satisfies $A\vec{x}=b$

$\vec{x_1}$ satisfies $A\vec{x}=0$

then $\vec{x_0}+\vec{x_1}$ satisfies $A\vec{x}=b$

$\Longleftrightarrow$

Sol'n $\{A\vec{x}+b\}$, if not empty, is an affine space modeled on the vector space sol'n: $\{A\vec{x}=0\}$

Differential Eq:

Ex. $\frac{d}{dt} x(t) = \lambda x(t) + g(t)$

$\uparrow$ a given function

rewrite the equation, to introduce operator "D" $\frac{d}{dt}$.

$(D-\lambda) \cdot x(t) = g(t) \quad (*)$

$\downarrow$ linear operator $\quad \downarrow$ ndim v.s.

If $\exists$ a sol'n $x_0(t)$ for eq. $(*)$
then $\exists$ a host of other sol'n by
adding to $\{ x_0(t) :$ sol'n of $(D-\lambda) x(t) = 0 \}$

$\therefore$ general sol'n $x(t) = x_0(t) + C \cdot e^{\lambda t}$.

$\uparrow$ free

General strategy to solve inhomogeneous eq:

1. Find a particular sol'n to the eq.

2. Find a general sol'n to the homoge eq.

3. Add up. get general sol'n to the inhom eq.

notation: $D = \frac{d}{dt}$.

Ex1: $D \cdot x(t) = C$

$\Rightarrow x_0(t) = C \cdot t$ is a particular sol'n

$D \cdot x(t) = 0$

$\Rightarrow x_1(t) = C_1$ is a general sol'n to homo version

$x(t) = x_0(t) + x_1(t) = Ct + C_1$

Ex2: $D \cdot x(t) = e^{\lambda t}$

particular sol'n: $x_0(t) = \frac{1}{\lambda} e^{\lambda t}$

gen sol'n to homo: $D \cdot x(t) = 0$.

$x_1(t) = C_0$.

gen: $x(t) = \frac{1}{\lambda} e^{\lambda t} + C_0$.

More generally.

Ex: $(D-\lambda)\, x(t) = g(t)$

suppose $g(t) = e^{\lambda_0 t}(b_0 + b_1 t + \cdots b_n t^n)$

① . $\lambda_0 \neq \lambda$

To find particular sol'n

we first set

$x(t) = e^{\lambda t} \cdot u(t)$

$$
\begin{aligned}
\text{LHS} &= (D-\lambda)\,(e^{\lambda t} \cdot u(t)) \\
&= e^{\lambda t} \cdot D\,u(t) + \lambda e^{\lambda t} \cdot u(t) - \lambda e^{\lambda t} u(t) \\
&= e^{\lambda t} \cdot D\,u(t)
\end{aligned}
$$

$$
\begin{aligned}
e^{\lambda t} \cdot D \cdot u(t) &= e^{\lambda_0 t} \cdot b(t) \\
D \cdot u(t) &= e^{(\lambda_0 - \lambda)t} \cdot b(t) \\
u(t) &= \int^t e^{(\lambda_0-\lambda)\tilde{t}}\, b(\tilde{t})\, d\tilde{t} \\
x(t) &= e^{\lambda t} \underline{\int^t e^{(\lambda_0-\lambda)\tilde{t}} \cdot b(\tilde{t})\, d\tilde{t}} \\
&= e^{\lambda t} \cdot P_n(t)
\end{aligned}
$$

($P_n(t)$ and $b(t)$ of same degree)

$$\int e^{\lambda t}\,dt = \frac{1}{\lambda} e^{\lambda t} + C.$$

$$\int e^{\lambda t}\cdot t\cdot dt = \int t\cdot d\left(\frac{e^{\lambda t}}{\lambda}\right)$$

integration by part $= t\cdot\frac{e^{\lambda t}}{\lambda} - \int \frac{e^{\lambda t}}{\lambda}\,dt$

$$= e^{\lambda t}\cdot\frac{t}{\lambda} - e^{\lambda t}\cdot\frac{1}{\lambda^2}$$

$$= \frac{e^{\lambda t}}{\lambda}\left(t - \frac{1}{\lambda}\right)$$

Check: $\left(e^{\lambda t}\frac{t}{\lambda} - e^{\lambda t}\frac{1}{\lambda^2}\right)' = \lambda\cdot e^{\lambda t}\cdot\frac{t}{\lambda} + e^{\lambda t}\cdot\frac{1}{\lambda}$

$$- \lambda e^{\lambda t}\cdot\frac{1}{\lambda^2}$$

$$= e^{\lambda t}\cdot t$$

In general, $\int e^{\lambda t}\cdot t^n\cdot dt = e^{\lambda t}\cdot \underbrace{P_n(t)}_{\text{polynomial of degree } n}$

② If $\lambda = \lambda_0$.

$(D-\lambda)\,x(t) = e^{\lambda t}\cdot b(t)$ polynomial

then set $x(t) = e^{\lambda t}\cdot u(t)$

we have $D\cdot u(t) = b(t)$

$$u(t) = \int b(t)\cdot dt$$

$$= P(t)$$

polynomial of deg $n+1$

$x(t) = e^{\lambda t} P(t)$ particular sol'n

See Gruental ODE book ex. 3.7.3 (b)(c)

Today :

① Fourier series: Method of expanding functions.

consider interval $[-L, L]$
consider complex valued (continuous, or Riemann integrable) functions on $[-L, L]$.

Let $f(x), g(x)$ be such nice functions on $[-L, L]$
we define an "inner product"

$$\langle f, g \rangle = \frac{1}{2L} \int_{-L}^{L} \overline{f(x)} \cdot g(x) \cdot dx$$

Compare this with Hermitian inner product on $\mathbb{C}^N$

$\vec{x} = (x_1 \cdots x_N)$, $\vec{y} = (y_1 \cdots y_N)$ $\in \mathbb{C}^N$.

$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{N} \overline{x_i} \cdot y_i$

think as discrete N points $\Rightarrow$ Continuous on functions

Ex

- $e_n(x) := e^{\frac{\pi i x}{L} \cdot n}$ complex valued. $n \in \mathbb{Z}$

- it is $2L$-periodic

$$e_n(x+2L) = e^{\frac{\pi i x}{L} \cdot n} \cdot \underbrace{e^{2\pi i \cdot n}}_{=1} = e_n(x)$$

$$\langle e_n(x), e_m(x) \rangle = \frac{1}{2L} \int_{-L}^{L} \overline{e^{\frac{\pi i}{L} x \cdot n}} \cdot e^{\frac{\pi i}{L} x \cdot m} dx$$

$$= \frac{1}{2L} \int_{-L}^{L} e^{\frac{\pi i}{L} x (m-n)} \cdot dx$$

$$= \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

Why 0: $\int_{-L}^{L} e^{\frac{\pi i}{L}\cdot k}\cdot dx$, $k = m-n \neq 0$

$$= \frac{e^{\frac{\pi i}{L}\cdot k\cdot x}}{\frac{\pi i}{L}\cdot k}\Big|_{-L}^{L}$$

$$= \frac{e^{\frac{\pi i}{L}\cdot L\cdot k} - e^{\frac{\pi i}{L}\cdot -L\cdot k}}{\frac{\pi i}{L}\cdot k}$$

$$= \frac{e^{k\pi i} - e^{-k\pi i}}{\sim}$$

$\because e^{\pi i} = -1$

$$= \frac{(-1)^k - (-1)^{-k}}{\sim}$$

$$= 0$$

Remark: If we consider "L^2 space on $[-L, L]$" with the above inner product. then $\{e_n(x)\}_{n\in\mathbb{Z}}$ forms an orthonormal basis

- Given any continuous function $f(x)$ on $[-L, L]$ we can define its complex Fourier coefficients $\{c_n\}_{n\in\mathbb{Z}}$.

$$c_n := \langle e_n, f\rangle = \frac{1}{2L}\int_{-L}^{L} e^{\frac{\pi i}{L}\cdot x\cdot n}\cdot f(x)\cdot dx$$

complex fourier series:

$$\sum_{n\in\mathbb{Z}} c_n\cdot e_n(x)$$

[Compare: suppose V a finite-dim. Hermitian v.s. and $\{e_1 \cdots e_n\}$ orthonormal basis. then for $\forall v\in V$,

$$v = \sum_{i=1}^{n} c_i e_i, \quad c_i = \langle e_i, v\rangle$$
]

Complex fourier expansion:

- For $f(x)$ continuous, then for $\forall x \in [-L, L]$, we have

$$f(x) = \underbrace{\sum_{n \in \mathbb{Z}} e_n(x)}_{\text{number given plugging in } x} \cdot \underbrace{\langle e_n, f \rangle}_{\text{number}} = \lim_{N \to \infty} \sum_{n=-N}^{N} e_n(x) \langle e_n, f \rangle$$

Real functions' fourier expansion

- Given a real valued function $f(x)$ on $[-L, L]$ we can use 1, $\sin(\frac{\pi x}{L} \cdot n)$, $\cos(\frac{\pi x}{L} \cdot n)$, $n = 1, 2, \cdots$ to expand f. i.e.

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cdot \cos\left(\frac{\pi x}{L} \cdot n\right) + b_n \cdot \sin\left(\frac{\pi x}{L} \cdot n\right)\right)$$

These 1, $\sin(\frac{\pi x}{L} \cdot n)$ and $\cos(\frac{\pi x}{L} \cdot n)$ forms an ONB

Notation: $S_n(x) = \sin(\frac{\pi x}{L}\cdot n)$ $\quad C_n = \cos(\frac{\pi x}{L}\cdot n)$

$$e_n(x) = e^{\frac{\pi i}{L}\cdot x\cdot n} = \cos(\frac{\pi}{L}\cdot x\cdot n) + i\sin(\frac{\pi}{L}\cdot x\cdot n)$$

complex conjugate

$$= C_n(x) + iS_n(x)$$

$$e_{-n}(x) = C_{-n}(x) + iS_{-n}(x)$$

$$= C_n(x) - iS_n(x)$$

$$C_n(x) = \frac{e_n + e_{-n}}{2}$$

$$S_n(x) = \frac{e_n - e_{-n}}{2i}$$

$$\langle C_n, S_n\rangle = \langle \frac{e_n + e_{-n}}{2}, \frac{e_n - e_{-n}}{2i}\rangle$$

$$= \frac{1}{4i}(\langle e_n, e_n\rangle - \langle e_{-n}, e_{-n}\rangle)$$

$$= \frac{1}{4i}(1-1) = 0$$

$$\langle C_n, C_n\rangle = \langle \frac{e_n + e_{-n}}{2}, \frac{e_n, e_{-n}}{2}\rangle$$

$$= \frac{1}{4}(\langle e_n, e_n\rangle + \langle e_n, e_{-n}\rangle)$$

$$= \frac{1}{4}\cdot 2$$

$$= \frac{1}{2}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n C_n(x) + b_n\cdot S_n(x))$$

To extract a_0, we do.

$$\langle 1, f(x)\rangle = \langle 1, a_0\rangle$$

$$\Downarrow$$

$$\frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$= \frac{1}{2L}\int_{-L}^{L} 1\cdot a_0 dx$$

$$= a_0.$$

$$\Rightarrow a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

To extract a_n we apply $\langle c_n, - \rangle$ $\quad c_n = \cos\left(\frac{\pi x}{L} \cdot n\right)$

$$\langle c_n, f(x) \rangle = a_n \langle c_n, c_n \rangle = \frac{1}{2} a_n$$

$$a_n = 2 \cdot \frac{1}{2L} \int_{-L}^{L} \cos\left(\frac{\pi x}{L} \cdot n\right) \cdot f(x) \cdot dx$$

Similarly,

$$b_n = \frac{1}{L} \int_{-l}^{L} \sin\left(\frac{\pi x}{L} \cdot n\right) f(x) \cdot dx.$$

Ex: let $f(x) = x$ on $[-L, L]$

find complex & Real fourier coefficients.

Read [ODE]

② Famous PDE

Given a function that depends on more than 1 variable say $f(x,y)$

we can talk about its partial derivative.

$$\partial_x f(x_0,y_0) := \lim_{\varepsilon\to 0} \frac{f(x_0+\varepsilon, y_0) - f(x_0,y_0)}{\varepsilon}$$

$$\partial_y f(x_0,y_0) := \lim_{\varepsilon\to 0} \frac{f(x_0,y_0+\varepsilon) - f(x_0,y_0)}{\varepsilon}$$

we can define high-order partial derivatives

$$\partial_x(\partial_x f), \qquad \partial_y(\partial_x f)$$

$$\partial_x(\partial_y f), \qquad \partial_y(\partial_y f)$$

In the case that f is "twice differentiable"

$\Rightarrow \quad \partial_x\partial_y f = \partial_y\partial_x f$

Ex: $\quad f(x,y) = x\cdot y.$

$\partial_x f = y. \qquad \partial_y f = x$

$\partial_x\partial_x f = 0 \qquad \partial_y\partial_y f = 0$

$\partial_x\partial_y f = \partial_y\partial_x f = 1$

Notation: $\quad f_x = \partial_x f \quad f_{xx} = \partial_x\partial_x f = \partial_x^2 f$

- Laplacian operator: Δ

on $\mathbb{R}^n \quad \Delta = (\partial x_1)^2 + (\partial x_2)^2 + \cdots (\partial x_n)^2$

- Δf measures how much a function "bend"

Ex1: on $\mathbb{R}$ $\quad \Delta f(x) = \partial_x^2 f(x)$

$\Delta f(x) = 0 \iff \partial_x^2 f(x) = 0.$

$\iff f(x) = a + bx$

straight line, no bending

Ex2: on $\mathbb{R}^2$ $\quad \Delta f(x,y) = \partial_x^2 f + \partial_y^2 f$

$\Delta f(x,y) = 0 \iff \partial_x^2 f(x,y) + \partial_y^2 f(x,y) = 0$

ex soln: $\quad f(x,y) = a + bx + cy$

$f(x,y) = xy$

or $f(x,y) = x^2 - y^2$

...

The sol'n to $\Delta f(x,y)=0$ are called harmonic functions

- Eigenvalue problem for Δ (laplace operator):

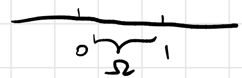
Ex: let $\Omega \subset \mathbb{R}^n$. $f: \Omega \to \mathbb{R}$

$$\begin{cases} \Delta f = \lambda f \\ f|_{\partial\Omega} = 0 \end{cases}$$

| λ: eigenvalue for Δ
| f: eigenvector

ex: $[0,1] \subset \mathbb{R}$ $f(x)$: s.t.

$$\begin{cases} \partial_x^2 f = \lambda f \\ f(0)=0 \quad f(1)=0. \end{cases}$$



in 1-dim we have the sequence of sol'n

$$\begin{cases} f_n(x) = \sin(\pi x \cdot n) \\ \lambda_n = -(\pi n)^2 \end{cases}$$

$$\partial_x f_n(x) = \partial_x \sin(\pi n \cdot x) = \cos(\pi n \cdot x)\cdot(\pi n)$$

$$\partial_x^2 f_n(x) = \partial_x \cos(\pi n \cdot x)(\pi n) = -\sin(\pi n \cdot x)(\pi n)^2$$

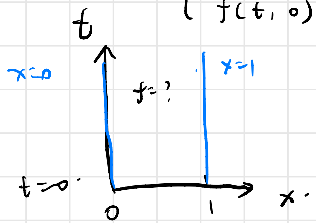
- Heat equation:

$$\partial_t f(t,x) = \Delta_x f(t,x) \qquad f: \underset{t}{\mathbb{R}} \times \underset{x}{\mathbb{R}^n} \to \mathbb{R}$$

in special case:

want $f: [0,\infty)_t \times [0,1]_x \to \mathbb{R}$.

s.t. $\begin{cases} \partial_t f(t,x) = \partial_x^2 f(t,x) \\ f(0,x) = f_0(x) & \text{initial} \\ f(t,0) = 0 \quad f(t,1) = 0 & \text{boundary} \end{cases}$



Fourier expansion

If $\quad f_0(x) = \sum_{n=1}^{\infty} b_n \cdot \sin(\pi x \cdot n)$

then we know how each $\sin(\pi x \cdot n)$ evolve.

i.e. if $f_0(x) = \sin(\pi x \cdot n)$

then $f(t,x) = e^{-(\pi n)^2 t} \sin(\pi x \cdot n)$

$$\partial_t f = -(\pi n)^2 \cdot e^{-(\pi n)^2 t} \cdot \sin(\pi x \cdot n)$$

$$\partial_{xx} f = e^{-(\pi n)^2 t} \cdot (-(\pi n)^2) \cdot \sin(\pi x \cdot n) \Rightarrow \partial_t f = \partial_{xx} f$$

then $f(t,x) = \sum_{n=1}^{\infty} b_n \cdot e^{-(\pi n)^2 t} \cdot \sin(\pi x \cdot n)$

Summary:

(1) $\frac{d}{dt}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = A \cdot \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$

A: constant coeff matrix.

sol'n: $\vec{x}(t) = e^{At} \cdot \vec{x}(0)$

where $e^{At} = I_n + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 \cdots$

if $A = C^{-1} J \cdot C$

$\quad\quad\quad\quad\llcorner$ Jordan form

then $e^{At} = C^{-1} \cdot e^{Jt} \cdot C$

- if A is diagonalizable.

$J = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$e^{Jt} = \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix}$

$\vec{x}(0)$ contains n free vairables

$x(0) \cdots \quad x_n(0)$

(2) Higher order const coeff eq.

$\left(\frac{d}{dt}\right)^n x(t) + a_1 \left(\frac{d}{dt}\right)^{n-1} x(t) + \cdots a_{n-1} \frac{d}{dt} x(t) + a_n x(t) = 0.$

Method 1: (general):

introduce new unknown functions;

$x_0(t) = x(t)$

$x_1(t) = \frac{d}{dt} x(t)$

$\vdots$

$x_{n-1}(t) = \left(\frac{d}{dt}\right)^{n-1} x(t)$

These new functions satisfy

$$\frac{d}{dt}\begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x^{(n)}(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ \vdots \\ x_{n-1}(t) \\ -a_1 x_{n-1} \cdots a_n \cdot x_0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & \\ 0 & & & & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_1 \end{pmatrix}\begin{pmatrix} x_0 \\ \vdots \\ \vdots \\ x_{n-1} \end{pmatrix}$$

this reduces to case (1)

Method 2: Find characteristic polynomial:

try to plug in

$x = e^{\lambda t}$

get equation

$$\underbrace{(\lambda^n + a_1\lambda^{n-1} \cdots + a_n)}_{= \det(\lambda - A)} \cdot e^{\lambda t} = 0$$

(3) solving PDE of type.

$\Delta_x u(t,x) = \partial_t u$, or $\partial_{tt} u$, ...

Heat: $\partial_t u(t,x) = \partial_{xx} u(t,x)$

Gen sol'n : separation of variables to obtain "basis" of sol'n :

$u(t,x) = f(t) \cdot g(x)$

then $\partial_t (f(t) \cdot g(x)) = \partial_{xx}(f(t) \cdot g(x))$

$\Longleftrightarrow (\partial_t f) \cdot g = f \cdot \partial_{xx} g$

$\Leftarrow$ if $f, g \neq 0$. $\dfrac{\partial_t f(x)}{f(x)} = \dfrac{\partial_{xx} g(x)}{g(x)} = \text{const}$

$\Rightarrow \begin{cases} \partial_t f(t) = \lambda f(t) & \text{for some } \lambda \\ \partial_{xx} g(x) = \lambda \cdot g(x) \end{cases}$

$\begin{cases} f(t) = e^{\lambda t} f(0) \\ g(x) = e^{\sqrt{\lambda} t} \cdot c_1 + e^{-\sqrt{\lambda} t} \cdot c_2 \end{cases}$