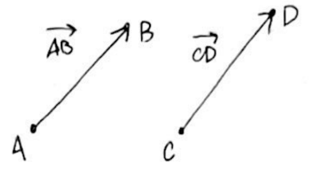
9/3/22

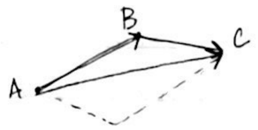
- Review Notes Givental ODE

1.1 Vectors

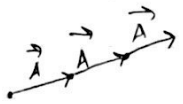
Def: A directed segment $\overrightarrow{AB}$ specified by an ordered pair of points



- $\overrightarrow{AB} = \overrightarrow{CD}$ if $|\overrightarrow{AB}| = |\overrightarrow{CD}|$ and obtained through translation



- Addition
  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{BC} + \overrightarrow{AB}$



- Scalar Multiplication
  $3\vec{A} = \vec{A} + \vec{A} + \vec{A}$

$\alpha\vec{u} + \beta\vec{v} + \ldots + \gamma\vec{w}$

- Linear Combinations
  - addition & Scalar Multiplication

1.1.2 Inner Product

$\langle \vec{u}, \vec{v} \rangle = |\vec{u}||\vec{v}| \cos\theta$

Properties.

a) Symmetric $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

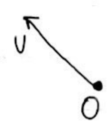
b) Nonzero vectors $\rightarrow \langle \vec{u}, \vec{u} \rangle > 0$

c) Bilinearity:

$$\langle \alpha\vec{u} + \beta\vec{v}, \vec{w} \rangle = \alpha\langle \vec{u}, \vec{w} \rangle + \beta\langle \vec{v}, \vec{w} \rangle$$
$$\langle \vec{w}, \alpha\vec{u} + \beta\vec{v} \rangle = \alpha\langle \vec{w}, \vec{u} \rangle + \beta\langle \vec{w}, \vec{v} \rangle$$

1.1.3 Coordinates.

Origin O & Directions of 2 $\perp$ axes



$\overrightarrow{Ou}$ - radius-vector

Orthonormal Basis:

e.g. $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

"orthonormal" - vectors are $\perp$

"basis" - any vector $\vec{u}$ can be uniquely written as a linear comb. of $\vec{e_1}, \vec{e_2}$

$$\vec{u} = u_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- More General Coord. Systems
  - pick any 2 non-proportional vectors $\vec{f_1}, \vec{f_2}$ on the role of a basis
    ↳ Given such f_1, f_2, any vector $\vec{u}$ is uniquely written as a linear comb. $\vec{u} = u_1\vec{f_1} + u_2\vec{f_2}$
  - New Inner Product:

$$\langle \vec{u}, \vec{v} \rangle = a u_1 v_1 + b u_1 v_2 + b u_2 v_1 + c u_2 v_2$$

* Although basis changes, inner product value doesn't change for same $\vec{u}$ & $\vec{v}$

where $a = \langle f_1, f_1 \rangle$
$b = \langle f_1, f_2 \rangle$
$c = \langle f_2, f_2 \rangle$

* Check!

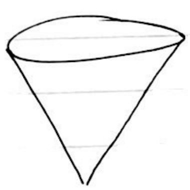
$$\langle u_1\vec{f_1} + u_2\vec{f_2}, v_1\vec{f_1} + v_2\vec{f_2} \rangle = \langle u_1\vec{f_1} + u_2\vec{f_2}, v_1\vec{f_1} \rangle + \langle u_1\vec{f_1} + u_2\vec{f_2}, v_2\vec{f_2} \rangle$$
$$= \langle u_1\vec{f_1}, v_1\vec{f_1} \rangle + \langle u_2\vec{f_2}, v_1\vec{f_1} \rangle + \langle u_1\vec{f_1}, v_2\vec{f_2} \rangle + \langle u_2$$
$$= u_1v_1\langle \vec{f_1}, \vec{f_1} \rangle + u_2v_1\langle \vec{f_2}, \vec{f_1} \rangle + u_1v_2\langle \vec{f_1}, \vec{f_2} \rangle + v_1u_1\langle$$

1.2 Analytical Geometry

1.2.1 Linear Func. & Straight Lines

Linear Func: $y = a_1x_1 + a_2x_2$
Graph in $\mathbb{R}^3$: $y = f(x_1, x_2)$
Levels/Level Curves: curves on the plane where given function of two vars. is constant

1.2.2 Conic Sections



* intersections of the plane w/ a circular conic surface in $\mathbb{R}^3$
* General Eq.:

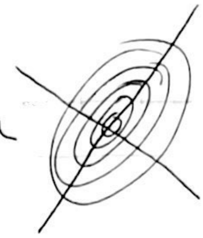
$$ax_1^2 + 2bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f = 0$$

Principal Axes - coordinate lines / symmetry axes of the normal ellipse

1.2.3 Quadratic Forms

$$\underbrace{ax_1^2 + 2bx_1x_2 + cx_2^2}_{\text{Quadratic Form}} + \underbrace{dx_1 + ex_2}_{\text{Linear Form}} + f = 0$$

Principal Axis:



- line through the origin in which the quadratic form is symmetric w/ respect to it

Thm: (Principal Axis Thm)

- Any quadratic form in 2 variables has two $\perp$ principal axes

Thm': (Orthogonal Diagonalization Thm)

- Any quadratic form in a suitable Cartesian coord. system takes on $AX_1^2 + CX_2^2$

Quadratic Eq. Transformations consist(s) of the following:

- Rotations of Coord. Systems
- Shift of the Origin
- Division of the eq. by a Constant

Corollary: Any quadratic form in suitable (not necessarily Cartesian) coordinate system assumes one of the normal forms:

$$X_1^2 + X_2^2,\ X_1^2 - X_2^2,\ X_1^2,\ X_2^2,\ 0$$

1.3 Linear Transformations & Matrices

1.3.1 Linearity

Def. A linear transformation T on a plane is a rule that to each vector $\vec{x}$ associates the vector $T\vec{x}$ on the same plane in a way that linear comb. of any vectors $\vec{x}, \vec{y}$ with coeff. α, β are transformed to linear comb. w/ same coeff.

$$T(\alpha\vec{x} + \beta\vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y}) \quad * \text{linearity}$$

ex. Rotation, Reflection (Orthogonal Transformation)

- preserve lengths & angles between vectors
- Thus... $\langle T\vec{x}, T\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$

• Uniquely determined by how it acts on basis:

$$T\vec{x} = T(x_1\vec{f_1} + x_2\vec{f_2})$$
$$= x_1T(\vec{f_1}) + x_2T(\vec{f_2})$$

Linear Transformation (linear form)

$$x_1' = ax_1 + bx_2 \qquad \begin{bmatrix} a \\ c \end{bmatrix} = T\vec{f_1}$$

$$x_2' = cx_1 + dx_2 \qquad \begin{bmatrix} b \\ d \end{bmatrix} = T\vec{f_2}$$

$$X' = TX \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = X'$$

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

* 2x2 matrix of the linear transform. T w/ respect to basis

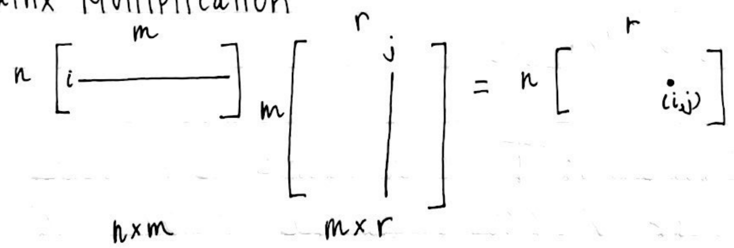
Types:

· Rotation $\quad T_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

· Reflection $\quad R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

· Orthogonal (rotation/ reflection)

• Matrix Multiplication



Properties:

(1) Not commutative

$AB \neq BA$ usually

Key Concepts:

· When changing basis, we are only changing how a vector is expressed not its inherent form

· Linear Transformation + Translation = "Affine" Linear Transformation

· $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$