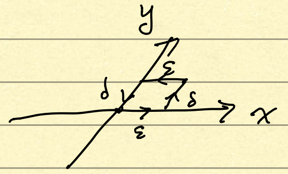
Recall last time:

- some formula for connections ∇ and curvatures F_∇.
- Geometric meaning of curvature for vector bundle.



E $r = \operatorname{rank} E$

$\downarrow$

$\mathbb{R}^2$

F_{12} component of F

$$F = F_{12} \cdot dx^1 \wedge dx^2, \quad F_{12}(x,y) \in M_r(\mathbb{R})$$

$$P_{(0,0)} = Id + \varepsilon \cdot \delta \cdot F_{12}(0,0) + \cdots$$

(as eqn in $M_r(\mathbb{R})$).

- Bianchi identity.

$$\begin{array}{c} End(E) = Hom(E,E) \\ \downarrow \\ M \end{array}$$

$$F \in \Omega^2(M, End(E)).$$

$\nabla^{End(E)}$ is a connection on $End(E)$. i.e. for any $T \in \Omega^0(M, End(E))$.

$$\nabla^{End(E)}(T) = \nabla^E \cdot T - T \cdot \nabla^E$$

i.e. for $u \in \Omega^0(M,E)$.

$$\left(\nabla^{End(E)}(T)\right)(u) = \nabla^E(Tu) - T(\nabla^E u)$$

in general: if $T \in \Omega^p(M, End(E))$.

$$|T| = p, \quad |\nabla| = 1.$$

$$\nabla^{End(E)}(T) = [\nabla^E, T] = \nabla^E \cdot T - (-1)^{|T|\cdot|\nabla|} \cdot T \cdot \nabla^E$$

- In case that. $T = F_\nabla \in \Omega^2(M, End(E))$

$$\Rightarrow [\nabla^E, F_\nabla] = \nabla^E \circ F_\nabla - (-1)^2 \cdot F_\nabla \cdot \nabla^E$$

$$= \nabla^E \cdot (\nabla^E \cdot \nabla^E) - (\nabla^E \cdot \nabla^E) \cdot (\nabla^E)$$

$$= 0.$$

$$\Rightarrow \nabla^{End(E)}(F_\nabla) = 0$$

(2) proof #2: work in local chart,

$$\nabla^E = d + A$$

$$F = \nabla^2 = d(A) + A \wedge A$$

[d: trivial connection on E w.r.t. the local trivialization.

$$= d(A) + \tfrac{1}{2}[A,A].$$

Now, we need

$$\nabla^{End(E)}(F) = [\nabla^E, F] = [d + A, dA + \tfrac{1}{2}[A,A]]$$

$$= \underbrace{d(d(A))}_{0 \ \because d^2 = 0} + \underbrace{d\left(\tfrac{1}{2}[A,A]\right) + [A, dA]}_{0} + \underbrace{\tfrac{1}{2}[A,[A,A]]}_{\text{Jacobi identity}}$$

1-form, const matrix

$$A \in \Omega^1(U, End(E)). \quad A = \alpha_i \otimes T_i$$

$$dA \in \Omega^2(U, End(E)) \quad dA = (d\alpha_i) \otimes T_i$$

Indeed: $[A,[A,A]]$ $\qquad A = \sum_i \alpha_i \otimes T_i$

$$= \sum_{i,j,k} \alpha_i \wedge \alpha_j \wedge \alpha_k \, [T_i, [T_j, T_k]]$$

T_i matrix const

α_i 1-form.

$$= \frac{1}{3} \sum_{i,j,k} \alpha_i \wedge \alpha_j \wedge \alpha_k \left\{ [T_i,[T_j,T_k]] + [T_j,[T_k,T_i]] + [T_k,[T_i,T_j]] \right\}.$$

$$= 0$$

$$\Rightarrow \nabla(F) = 0. \quad \text{(Bianchi identity).}$$

Today: connection on tangent bundle.

$$E = TM.$$

- Definition: torsion tensor. T.

given a connection ∇ on TM,

$$T: Vect(M) \times Vect(M) \to Vect(M)$$

$X, Y \in Vect(M)$

(1) $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.

(2) $\forall f \in C^\infty(M)$

$$T(fX, Y) = f\, T(X,Y).$$

indeed: $T(fX,Y) = \nabla_{fX} Y - \nabla_Y(fX) - [fX, Y]$

$$= f \cdot \nabla_X Y - f \nabla_Y(X) - Y(f) \cdot X - f[X,Y] + Y(f)X$$

$\therefore\ T \in \Omega^2(M, TM)$. torsion.

- In local chart, given local coords $x^1, \cdots, x^n$ on $\mathcal{U}$, $\exists\ \{\partial_{x^1}, \cdots, \partial_{x^n}\}$ on $TM|_{\mathcal{U}}$.
  $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ``\partial_1 \qquad ``\partial_n$

$$\nabla_{\partial_i}(\partial_j) = \underset{\sim}{\Gamma^k_{ij}}\,\partial_k$$

recall $\nabla_{\partial_i}(e_\alpha) = \Gamma^\beta_{\underline{i}\alpha}\, e_\beta$

Γ^k_{ij} is the Christofell symbol (??).

- $T(\partial_i, \partial_j) = T^k_{ij}\,\partial_k$ (sum over k)

$$\begin{aligned} T(\partial_i, \partial_j) &= \nabla_{\partial_i}(\partial_j) - \nabla_{\partial_j}(\partial_i) - \underbrace{[\partial_i, \partial_j]}_{=0} \\ &= \Gamma^k_{ij}\,\partial_k - \Gamma^k_{ji}\,\partial_k \\ &= (\Gamma^k_{ij} - \Gamma^k_{ji})\,\partial_k \end{aligned}$$

so torsion tensor measure the a-symmetry of Γ^k_{ij} in i, j indices.

If $T = 0$, we say ∇ is torsionless or symmetric.

Levi-Cevita connection:

- Given a g Riemannian metric. There exist a unique connection ∇, s.t.

$$T_\nabla = 0, \quad \nabla(g) = 0.$$

$\underset{\sim}{g}$. the metric tensor.

$$g(X, Y) \in C^\infty(M).$$

$$g \in \Gamma(M, T^*M \otimes T^*M).$$

$\underline{\nabla}$ is a connection on TM.

$\rightsquigarrow$ - - on T^*M

- - - - on $T^*M \otimes T^*M$

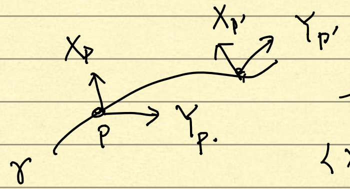
$\underset{\sim}{\nabla(g)} = 0$

$\Leftrightarrow\ \forall\ X, Y, Z \in \mathrm{Vect}(M)$.

$$\nabla_Z(g(X, Y))$$

$$\overset{\text{always}}{\underset{\text{true.}}{=}}\ (\underset{\sim}{\nabla_Z g})(X, Y) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

$$\boxed{\begin{aligned} Z(g(X,Y)) = g(\nabla_Z X, Y) \\ + g(X, \nabla_Z Y). \end{aligned}}$$



then.

$$\langle X_p, Y_p \rangle = \langle X_{p'}, Y_{p'} \rangle.$$

i.e. parallel transport preserves inner product.

(4.1.3) [Ni].

$$\begin{aligned} g(\boxed{\nabla_X Z}, Y) = \tfrac{1}{2}\{ & X g(Y, Z) - Y(g(Z, X)) \\ & + Z g(X, Y) - g([X, Y], Z) \\ & + g([Y, Z], X) - g([Z, X], Y)\}. \end{aligned}$$

more useful formula in Christofell symbol:

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{il}(-\partial_l g_{jk} + \partial_j g_{kl} + \partial_k g_{jl})$$

notice it is symmetric in $j \leftrightarrow k$.

$$\Gamma_{i,jk} := g(\nabla_{\partial_j}\partial_k, \partial_i) \qquad \Gamma^i_{jk} = g^{il}\cdot\Gamma_{l,jk}.$$

$$\nabla_{\partial_j}(\partial_k) = \nabla_{\partial_k}(\partial_j) \quad (\because T = 0).$$