- M : sm mfd. $\quad E \to M$ rank r v.b.

∇ connection on E:

$\Gamma(M,E) = C^\infty(M,E)$

$$\nabla : \Omega^0(M,E) \longrightarrow \Omega^1(M,E)$$

s.t. $\forall\ s \in \Omega^0(M,E)$, $f \in C^\infty(M)$.

$$\nabla(fs) = \nabla(f)\cdot s + f\cdot\nabla(s)$$
$$= df \otimes s + f\cdot\nabla(s).$$

∇ is called exterior covariant derivative.

∇_X, for X a vect field, is covariant derivative

- $\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M,E)$.

- Some useful (coord-free) formula:

X : v.f.

$\nabla,\ \nabla_X,\ \iota_X,\ \mathcal{L}_X.$

$\deg(\nabla) = 1 \qquad \deg(\nabla_X) = 0$

$\deg(\iota_X) = -1 \qquad \deg(\mathcal{L}_X) = 0.$

(1) Generalized Leibniz rule:

$\omega \in \Omega^k(M)$, $\quad u \in \Omega^p(M,E)$

$\omega \otimes u = \omega \wedge u \in \Omega^{k+p}(M,E).$

$$\nabla_X(\omega\wedge u) = \mathcal{L}_X(\omega)\wedge u + \underbrace{(-1)^{|\nabla_X|\cdot|\omega|}}_{=1}\cdot\omega\wedge\nabla_X(u).$$

$\nabla_{\_}$ is like d.

∇_X is like $[\iota_X, \nabla]$.

- $\nabla_X = [\iota_X, \nabla] = \iota_X\cdot\nabla - (-1)^{|\iota_X|\cdot|\nabla|}\cdot\nabla\circ\iota_X$
$= \iota_X\cdot\nabla + \nabla\circ\iota_X$

- $[\iota_X, \iota_Y] = 0, \qquad \iota_X\iota_Y + \iota_Y\iota_X = 0.$

...

$$F_\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \quad \in \Omega^0(M, \mathrm{End}(E))$$

(1) local form: on a small open $U \subset M$.

say $x^1, \cdots, x^n$ are local coord on M

$e_1, \cdots, e_r$ are local frame on E.

then on U

$\nabla = d + A \qquad A \in \Omega^1(M, \mathrm{End}(E)).$

$F = \nabla^2 = (d+A)^2 = d^2 + dA + Ad + A\wedge A$

$= d(A) + A\wedge A.$

Q: $A\cdot d$ terms is gone?

section $\cdot$ $A \in \Omega^1(U, \mathrm{End}(A)) \simeq \Omega^1(U)\otimes M_n(\mathbb{R})$

operation $\cdot$ $\Phi_A := A\wedge(\cdot) : \Omega^0(U,E) \to \Omega^1(U,E).$

$$\nabla = d + \Phi_A = d + A\wedge(\cdot)$$
$$\nabla^2 = (d+\Phi_A)\circ(d+\Phi_A)$$
$$= d^2 + \Phi_A\circ\Phi_A + d\cdot\Phi_A + \Phi_A\cdot d.$$

$$d\Phi_A + \Phi_A d = [d, \Phi_A] = \Phi_{dA}.$$

if $A = \omega \otimes u$ (1-form ω, section u)

$dA \in \Omega^2(U, \mathrm{End}(E)).$ $\qquad dA = (d\omega)\otimes u.$

But, by convention:

$d(A) = d\circ A + A\circ d.$

$\Phi_A\circ\Phi_A$, $\quad A = \sum_{i=1}^n dx^i \otimes \Gamma_i \qquad dx^i$

$\Gamma_i \in \Omega^0(U, \mathrm{End}(E))$

$C^\infty(M_n(\mathbb{R}))$

(1) $A\wedge A = \sum_{i,j} dx^i\wedge dx^j \otimes \underbrace{(\Gamma_i\circ\Gamma_j)}$.

composition of or matrix multiplication

$$= \sum_{i<j} dx^i\wedge dx^j \otimes (\Gamma_i\circ\Gamma_j - \Gamma_j\circ\Gamma_i)$$

(2) $[A, A]$

if $\mathfrak{g}$ is a Lie algebra.

and $\eta \in \Omega^k(M, \mathfrak{g})$

$\omega \in \Omega^p(M, \mathfrak{g})$. $\quad \Omega^k(M) \quad \mathfrak{g}$

$[\eta, \omega]$, $\quad$ say $\eta = \sum_{i\in I} \eta_i \otimes A_i$

$\omega = \sum_{j\in J} \omega_j \otimes B_j$

$$[\eta, \omega] = \sum_{\substack{i\in I\\ j\in J}} (\eta_i\wedge\omega_j)\otimes[A_i, B_j].$$

Claim: $[A, A] = 2\cdot A\wedge A.$

$d(A) \neq d\circ A$

$d(A) = d\circ A + A\circ d.$

$$F = d(A) + A\wedge A$$
$$= dA + \tfrac{1}{2}[A, A].$$

$\Gamma(U, \mathrm{End}(E)).$

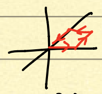
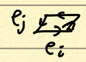
$$A = \sum_{i=1}^n dx^i \otimes \Gamma_i$$

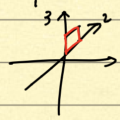
$$\Gamma_i = \sum_{\alpha=1}^r \sum_{\beta=1}^r \Gamma^\alpha_{i\beta}\, \underbrace{e_\alpha\otimes\delta^\beta}_{\in E\otimes E^* = \mathrm{End}(E)}$$

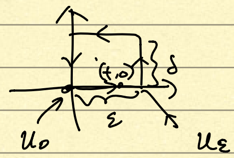
$$F = (d+A)\cdot(d+A)$$
$$= \sum_{i<j} F_{ij}\ dx^i\wedge dx^j$$

$$\boxed{F_{ij} = \partial_i\Gamma_j - \partial_j\Gamma_i + \Gamma_i\Gamma_j - \Gamma_j\Gamma_i}$$

- Geometrically, the curvature describe the failure to return to original vector, as you parallel transport around a loop.

i.e. F_{12}  

F_{23} 

Consider on $\mathbb{R}^2$, 

$$\underset{\displaystyle F_{12}(0)}{\overset{M_r(\mathbb{R})}{\Downarrow}} = \lim_{\substack{\varepsilon\to 0\\ \delta\to 0}} \frac{1}{\varepsilon}\frac{1}{\delta}\left[P_{\square} - Id\right].$$

say, we have $u_0 \in E_{(0,0)}$

$$\frac{\partial u_t}{\partial t} = -\underbrace{\Gamma_1((t,0))}_{\text{matrix, } r\times r.}\ \underbrace{u_t}_{\text{column vector } \left(\vdots\right)\}r.}$$

$t\in[0,\varepsilon]$. Keep upto ε term.

approx Γ_1 to be constant, then

$$u_\varepsilon = \underbrace{(1-\varepsilon\cdot\Gamma_1(0,0))}_{P_\rightarrow}\cdot u_0 + O(\varepsilon^2)$$

$P_{\square}$

$$P_\downarrow\cdot P_\leftarrow\cdot P_\uparrow \circ P_\rightarrow = (1+\delta\cdot\Gamma_2)(1+\varepsilon\Gamma_1(0,\delta))$$
$$\cdot(1-\delta\Gamma(\varepsilon,0))(1-\varepsilon\Gamma_1).$$
$$= 1-\varepsilon\cdot\delta\cdot\left(\partial_1\Gamma_2-\partial_2\Gamma_1+[\Gamma_1,\Gamma_2]\right)\Big|_{(0,0)}$$

$$\therefore\ F_{12}(0)\cdot u = \lim_{\substack{\varepsilon,\delta\\ \to 0}}\frac{1}{\varepsilon\cdot\delta}\left(P_{\square}\cdot u - u\right).$$

$u\in E_{(0,0)}$.