Cohomology:

① compactly supp cohomology

② Poincaré duality.

③ Mayer-Vietoris sequence. (SES $\to$ LES).

④ Čech cohomology (sheaf cohomology).

- De Rham complex: M sm mfd. $\dim M = n$.

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \cdots \to \Omega^n(M).$$

Aside: · given a p-form ω on M, and a smooth cpt p-dim sub mfd $\iota: S \hookrightarrow M$, we can do integration.

$$\int_S \omega = \int_S \iota^* \omega \quad \left(\begin{array}{c} \text{p-form } \iota^*\omega \\ \text{on a p-dim } \cancel{\text{sub}}\text{mfd } S \end{array} \right).$$

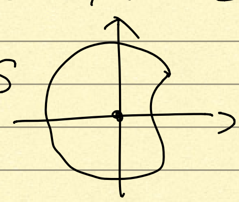
· if ω is a closed p-form, then

$$\int_S \omega \quad \text{is invariant under "deformation" of } S.$$

$\omega \in \Omega^1(\mathbb{R}^2 \setminus 0)$. $\partial S = \phi$. S cpt.

Ex. $$\int_S \underbrace{\frac{x\,dy - y\,dx}{x^2+y^2}}_{\omega}$$

S



is indep of perturbation of S.

- compactly support de Rham:

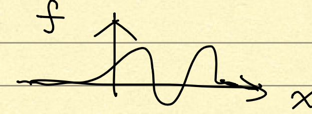
· we say $\omega \in \Omega^p(M)$ is compactly supported, if $\exists K \subset M$ compact set, such $\omega\big|_{M \setminus K} = 0$

· (the condition is vacuous, if M is cpt itself).

$$\Omega_c^0(M) \xrightarrow{d} \Omega_c^1(M) \xrightarrow{d} \Omega_c^2(M) \to \cdots$$

($\because$ if ω is cptly supp, then $d\omega$ is also cptly supp.).

$$H_c^*(M) = H^*(\underline{\Omega_c^*(M)}, \underline{d}).$$

• **Ex:** $M = \mathbb{R}$. cptly supp fcn: 

• so: if f is cptly supp. and $df = 0$.

$\Rightarrow f(x) = 0$. $\forall x \in \mathbb{R}$.

$\therefore \quad H^0_c(\mathbb{R}) = Z^0(\Omega^*_c(\mathbb{R})) = 0.$

$$H^1_c(\mathbb{R}) = \frac{Z^1_c(\mathbb{R})}{B^1_c(\mathbb{R})} = \frac{\{\omega \in \Omega^1_c, \cancel{d\omega = 0}\}}{\{df : f \in \Omega^0\}}.$$

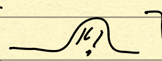
$\omega = g(x)\,dx.$ $\quad g \in C^\infty_0(\mathbb{R}).$

$$\int_{-\infty}^{+\infty} \omega = \int_{-\infty}^{+\infty} g(x)\,dx \qquad \int_{-\infty}^{+\infty} df = f(x)\Big|_{-\infty}^{+\infty} = 0$$

Claim: $\omega \in B^1_c(\mathbb{R})$, iff $\int_{-\infty}^{+\infty} \omega = 0$

indeed: we can define $f(x) = \int_{-\infty}^{x} \omega$

$H^1_c(\mathbb{R}) \simeq \mathbb{R}.$

$[$$] \mapsto 1$

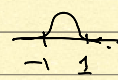
$\omega = g(x)\,dx$

$\int_{-\infty}^{+\infty} g(x)\,dx = 1.$

> **Stokes Thm:**
> $\int_S d\eta.$
> $\overset{!!}{}$
> • $\langle S, d\eta \rangle = \langle \partial S, \eta \rangle.$
> S: k-mfd.
> η: $(k-1)$-form.

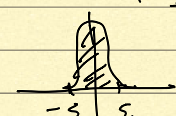
• $M = \mathbb{R}$, $\quad S = \{0\} \subset \mathbb{R}$.

Goal: represent S using diff form: a bump form situated at S. Fix a positive function $\rho(x)$, s.t. $\operatorname{supp}(\rho) \subset [-1,1]$.

$$\int_{-1}^{1} \rho(x)\,dx = 1$$ 

want have

$$\omega_{S,\varepsilon} := \rho_\varepsilon(x)\,dx.$$

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right).$$ 

• $\mathbb{R}^n$, $S = \{0\}$, $\omega_{S,\varepsilon} = \rho_\varepsilon(x_1)\cdots\rho_\varepsilon(x_n)\,dx_1\cdots dx_n.$

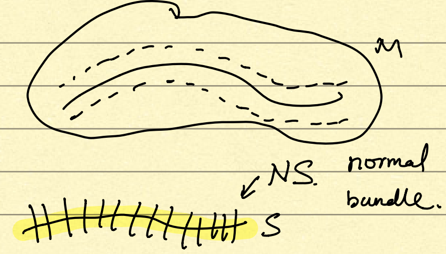
bump form at 0, $\int_{\mathbb{R}^n} \omega = 1$

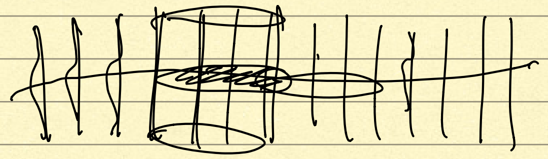
• M^n ~~gen~~ sm mfd., $\quad S^k \subset M$ cpt sm. submfd.

· $\mathcal{U}(S) \simeq V \subset NS$

tubular nbhd around S

↑ nbhd of zero section.



· 

$$\left(0 \to TS \to TM|_S \to NS \to 0 \right)$$

$$NS = \frac{TM|_S}{TS}.$$

· then we produce ω_S, to be a $(n-k)$-form,

"along the $\underbrace{\text{fiber of the normal bundle}}_{\simeq \mathbb{R}^{n-k}.}$".

① concentrated near S

② restriction to each normal slice. is a bump-form of $(n-k)$-deg.

· we have a way to go from cpt sm $\overbrace{\text{submfd}}^{\text{of dim } k}$ to.
$\Omega_c^{n-k}(M)$.

· <u>Poincaré Duality</u>:

$$H^k(M) \otimes H_c^{n-k}(M) \longrightarrow H_c^n(M) \xrightarrow{\int_M} \mathbb{R}.$$

$$([\alpha],[\beta]) \longmapsto \int_M \alpha \wedge \beta.$$

this pairing is a perfect pairing.

$p: V \times W \to \mathbb{R}$.
if $v \in V$ $v \neq 0$
$\exists\, w \in W$,
$p(v,w) \neq 0$.

· $\dim V = n$
$\Lambda^k V \times \Lambda^{n-k} V \to \Lambda^n V \simeq \mathbb{R}$

<u>Ex</u>:
$\mathbb{R}$

$H^0 \cong \mathbb{R}$. $\qquad H_c^0 = 0$

$H^1 = 0. \qquad H_c^1 = \mathbb{R}$.

$$\dim H^k = \dim H_c^{n-k}.$$

· $S \subset M$, k-dim submfd. . cpt.
$\omega \in \Omega^k(M)$

$$(\underset{k}{[\omega]}, \underset{n-k}{[\omega_S]}) = \int_S \omega.$$

$\parallel$

$$\int_M \omega \wedge \omega_S$$

(Remark: need to consider orientation).

- MV sequence: $M = U \cup V$.

<u>Question</u>: compute $H^*(M)$ using $H^*(U)$, $H^*(V)$, $H^*(U\cap V)$?

- $\forall\, k \in \{0, \cdots, n\}$.

SES.
$$(\alpha, \beta) \longmapsto \alpha|_{U\cap V} - \beta|_{U\cap V}$$

$$0 \to \Omega^k(M) \xrightarrow{p} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{q} \Omega^k(U\cap V) \to 0$$
$$\alpha \mapsto (\alpha|_U, \alpha|_V)$$
by restriction.

exact means:
- $\ker(p) = 0$
- q is surjective.
- $\ker(q) = \mathrm{im}(p)$.

- SES $\to$ LES:

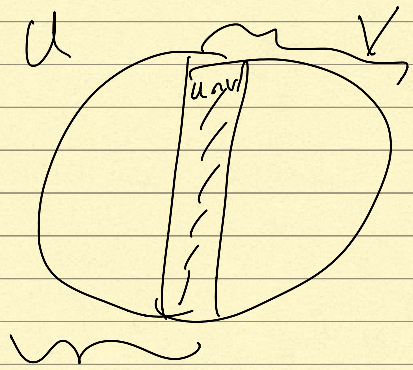
$$\begin{array}{ccccccccc} & & A^2 & & B^2 & - & C^2 & \to & \cdots \\ & & \uparrow & & & & & & \\ 0 & \to & A^1 & \to & B^1 & \to & C^1 & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & A^0 & \to & B^0 & \to & C^0 & \to & 0 \end{array}$$

then one can take cohomology in each column.

$$\cdots$$
$$\to H^1(A) \to H^1(B) \to H^1(C) \to$$
$$\delta$$
$$0 \to H^0(A) \to H^0(B) \to H^0(C) \to$$

MV seq:
$$\to H^1(M) \to H^1(U) \oplus H^1(V) \to \cdots$$
$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U\cap V) \to$$

<u>Ex</u>: compute $H^*(S^n)$. $\qquad S^n = D^n \underset{S^{n-1}}{\cup} D^n$



$[\alpha] \to ?$

$$H^k(U\cap V) \to H^{k+1}(M)$$

$\alpha \in \Omega^k(U\cap V)$, $d\alpha = 0$.

$\rho_U + \rho_V = 1$

$d(\alpha \cdot \rho_U) \qquad \alpha \cdot \rho_V$