- Cohomology : · ref: Lee & Nicolascu.

  - De Rham Cohomology
  - Singular Cohomology.
  - CW-complex, CW-cohomology.
  - Čech cohomology.
  - Morse cohomology.

- Cohomology (with coefficient in $\mathbb{R}$) is a "machine", with <u>input</u> a sm manifold (+ auxillary structure) and <u>output</u> a graded vector space over $\mathbb{R}$.

$$M \rightsquigarrow H^*(M) \qquad H^0(M) \oplus H^1(M) \oplus \cdots \oplus H^n(M).$$

- <u>functoriality</u>: "<u>contra</u>variant"

$$\begin{pmatrix} N \\ f \downarrow \\ M \end{pmatrix} \rightsquigarrow \begin{pmatrix} H^*(N) \\ \uparrow f^* \\ H^*(M) \end{pmatrix} \quad \text{"pull back map"}$$

- <u>De Rham Cohomology</u>: $H^*_{dR}(M)$.

  - M: sm n-dim mfd. · $\Omega^p(M) := \{\text{differential } k\text{-forms on } M\}$
    - linear space over $\mathbb{R}$
    - (· ∞-dim).

  - exterior derivative: $d : \Omega^p(M) \to \Omega^{p+1}(M)$

$$d^2 = 0.$$

$\rightsquigarrow$ "cochain complex" :

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \cdots \to \Omega^n(M)$$

$$d^2 = 0$$

- kernel:

$$Z^p = \{\omega \in \Omega^p(M) \mid d\omega = 0\}$$

$$B^p = \{\omega \in \Omega^p(M) \mid \exists \eta \in \Omega^{p-1}(M),\ d\eta = \omega\}.$$

$\because d^2 = 0 \quad \therefore$ if $\omega = d\eta$, then $d\omega = d(d\eta) = 0$.

$\therefore \quad B^p \subset Z^p$

$$\therefore \quad H^p_{dR}(M) := \frac{Z^p}{B^p}. \qquad \left(\text{Definition of dR cohomology.}\right)$$

this turns out to be finite dim vector space.

- Example: (1) $\mathbb{R}$, $\Omega^0 =$ sm fcn, $\Omega^1 = f(x)\,dx$

$$Z^0 = \{ f \mid df = 0 \} = \text{constant functions.}$$

$$B^0 = \mathrm{Im}(d : \Omega^{-1} \to \Omega^0) = \{0\}.$$

☆ · $H^0 = Z^0 / B^0 = Z^0 \simeq \mathbb{R}$

· $H^1 = Z^1 / B^1$.

· $Z^1 = \{ f(x)\,dx \mid d\cdot(f(x)\,dx) = 0 \} = \Omega^1(\mathbb{R})$

$B^1 = \{ f(x)\,dx \mid \exists F(x).\ dF = f(x)\,dx \} = \Omega^1(\mathbb{R})$

☆ $H^1(\mathbb{R}) = 0$.

(2) $\mathbb{R}^n$. $\Omega^p = \{ \sum_{i_1 < \cdots < i_p} \omega_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} = \sum_{\substack{I \\ |I| = p}} \omega_I \, dx^I \}$.

$H^0 \cong \mathbb{R}$.

$H^1 = ?$ (closed). $Z^1 = \{ \sum \omega_i dx^i \mid d(\sum \omega_i dx^i) = 0 \}$ (close 1-fom)

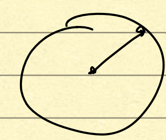
$= \{ \sum_i \omega_i dx^i \mid \partial_j \omega_i = \partial_i \omega_j, \ \forall i, j \}$.

(exact) $B^1 = \{ dF = \partial_i F \cdot dx^i \}$. (exact 1-form)

· (Poincaré Lemma): on $\mathbb{R}^n$, every closed 1-form is exact.

Pf: given ω closed 1 form, we construct F fun.

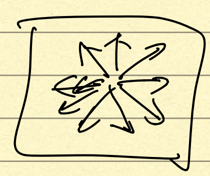
by $F(p) = \int_0^p \omega$. line integral.



· (Poincaré Lemma for p-form): say ω is a p-form, and $d\omega = 0$, we need to construct a $(p-1)$-form η.

s.t. $\omega = d\eta$. We use a Euler vector field:

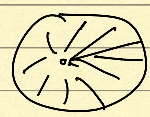
$$E = \sum_i x^i \frac{\partial}{\partial x^i}$$



· η is obtained by integrate ω along E. (see Lee for details).

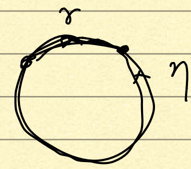
$\Rightarrow$ $Z^p = B^p$ for $p \geq 1$. $H^p(\mathbb{R}^n) = 0$. $(p \geq 1)$

(in general, true for any open sets $U \subset \mathbb{R}^n$, that is contractible.)



(3). · S^1 : $H^0(S^1) \simeq \mathbb{R}$

$H^1(S^1) \simeq \mathbb{R}$.



1-form $\eta =$ "$d\theta$" $\in \Omega^1(S^1)$

$\eta \in Z^1(S^1)$

$[\eta]$ the quotient image of η in Z^1/B^1

cohomology class that η belongs.

$$[\eta] \neq 0 \iff \int_{S^1} \eta \overset{\text{"}2\pi\text{."}}{\neq} 0.$$

· S^n : $H^0(S^n) \cong \mathbb{R}$

$H^p(S^n) \cong 0 \qquad 0 < p < n$

$H^n(S^n) \simeq \mathbb{R}$.

volume form : Vol_{S^n} is a top dim form.

$$\int_{S^n} Vol_{S^n} = vol(S^n) \neq 0.$$

$[Vol_{S^n}] \neq 0$.

In general : $\begin{cases} \cdot \text{ if } M \text{ is conn.} & H^0(M) \simeq \mathbb{R}. \\ \cdot\cdot \text{ if } M \text{ is compact,} & H^n(M) \simeq \mathbb{R}. \end{cases}$

· $\underline{M \times N}$. $\quad H^*(M \times N) = \underbrace{H^*(M) \otimes H^*(N)}$

tensor product of (graded) vector space.

$$H^k(M \times N) \simeq \bigoplus_{i+j=k} H^i(M) \otimes H^j(N).$$

· n-torus $T^n = (S^1)^n$.

$$H^i(T^n) = \begin{cases} \mathbb{R} & i = 0 \\ \mathbb{R}^{\binom{n}{i}} & 1 < i < n \\ \mathbb{R} & i = n \end{cases}$$

• Prop: if $M \underset{G}{\overset{F}{\rightrightarrows}} N$ are 2 smooth maps that are homotopic.

then. $H^*(N) \underset{G^*}{\overset{F^*}{\rightrightarrows}} H^*(M)$ and $F^* = G^*$. $\left(\begin{matrix} F_t & t\in[0,1] \\ & F_0 = F \\ & F_1 = G \end{matrix}\right)$

• More generally, if $\omega \in Z^p(N)$, then $\exists\, \eta \in \Omega^{p-1}(M)$

$$F^*\omega - G^*\omega = d\eta.$$

The way to prove such η exists, is to construct a "homotopy operator": for $F^* - G^*$.

$$h: \Omega^p(N) \to \Omega^{p-1}(M).$$

Satisfies: $\left([d,h] = \right) \boxed{d\circ h + h\circ d \overset{*}{=} (F^* - G^*)}$

$d\cdot h = (-1)^{|\cdot|}\, h\cdot d$

then indeed
$$\begin{aligned} (F^* - G^*)(\omega) &= (d\circ h + h\circ d)(\omega) \\ &= d\circ(h\omega) + h(\cancel{d\omega}) \quad 0 \because \omega \in Z^p(\cancel{M}). \\ &= \underset{\eta.}{d(\underset{\shortparallel}{h\omega})} \end{aligned}$$

let

• $H: \underline{M \times I} \to N$ be the homotopy between F, G.

$h = \left(\begin{matrix}\text{integrate} \\ \text{along } I\end{matrix}\right) \circ \underline{H^*}.$ $\quad \begin{matrix} M\times I \\ \downarrow \pi. \\ M \end{matrix}$

• Cor: if M, N are homotopic.

$$\begin{tikzcd} M \arrow[r, bend left, "f"] & N \arrow[l, bend left, "g"] \end{tikzcd} \quad \text{s.t.} \quad \begin{tikzcd} M \arrow[r, bend left, "g\circ f"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow"] & M \end{tikzcd}$$

$$\begin{tikzcd} N \arrow[r, bend left, "f\circ g"] \arrow[r, bend right, "id_N"'] \arrow[r, phantom, "\Downarrow"] & N \end{tikzcd}$$

then $H^*(M) \underset{f^*}{\overset{g^*}{\simeq}} H^*(N).$

$$\underset{\text{gp}}{H^1(M)} \simeq \operatorname{Hom}(\underset{\text{gp.}}{\pi_1(M)}, \underset{\text{ab.}}{\mathbb{R}})$$