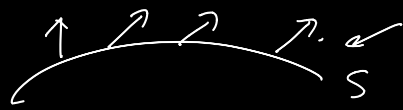
<u>Today</u>: (1). First, Second variation formula.

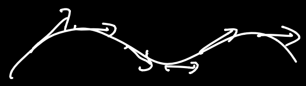
(2). Jacobi equation (for family of geodesics).

- (M,g). $S \subset M$ submfd.
- $TM|_S = \{(x,\xi) \in TM \mid x \in S,\ \xi \in T_xM\}$.
- $\Gamma(S, TM|_S)$:



S ← picture for Y.

$\Gamma(S, TS)$:



S ← picture for X

- ∇^M, ∇^S.

Let $X \in \Gamma(S, TS)$, $Y \in \Gamma(S, TM|_S)$

$\mathcal{U}$ nbhd of S

S

$\boxed{\text{How to define } \nabla^M_X Y?}$

<u>method ①</u> : extend X, Y to

$\tilde{X} \in \Gamma(\mathcal{U}, TM)$ $\quad \tilde{X}|_S = X$

$\tilde{Y} \in \Gamma(\mathcal{U}, TM)$. $\quad \tilde{Y}|_S = Y$.

$$\nabla^M_X Y := \left(\nabla^M_{\tilde{X}} \tilde{Y}\right)\Big|_S \in \Gamma(S, TM|_S)$$

Γ doesn't dep on extension.

say $\tilde{X}_1$, $\tilde{X}_2$ both extends X,

$$(\tilde{X}_1 - \tilde{X}_2)|_S = 0.$$

$$\left(\nabla^M_{(\tilde{X}_1 - \tilde{X}_2)}(\tilde{Y})\right)^j\Big|_S = \underbrace{(\tilde{X}_1 - \tilde{X}_2)^i}_{\to 0}\left[\partial_i(\tilde{Y}^j) + \Gamma^j_{ik}\cdot \tilde{Y}^k\right]\Big|_S$$

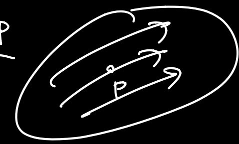
$$= 0.$$

similarly, if $\tilde{Y}_1$, $\tilde{Y}_2$ both extends Y,

$$\left(\nabla^M_{\tilde{X}}(\tilde{Y}_2 - \tilde{Y}_1)\right)\Big|_S = 0.$$

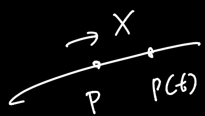
<u>method ②</u> : use parallel transport.

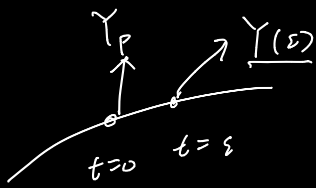
$\cdot \ (\nabla_X Y)\big|_p = \lim_{\varepsilon \to 0} \dfrac{P_{p(\varepsilon)\rightsquigarrow p}(Y_{p(\varepsilon)}) - Y_p}{\varepsilon}$

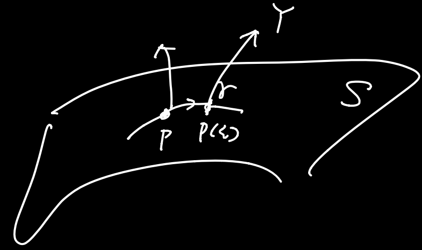


X

$p(t) = \Phi^X(t)(p)$.





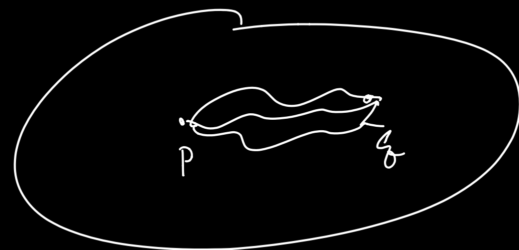


∘ Given X on S

we use that to generate little path, do parallel transport along those path.

∘ Variation Formula :

$E = \frac{1}{2}\int_0^1 |\dot\gamma(t)|_g^2 \, dt$



(for simplicity, assume γs are smooth path).

$\Omega_{p,q}$: path

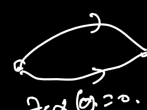
∘ 1st variation :

1-param family $\alpha_s(t)$, $s \in (-\varepsilon, \varepsilon)$

$$\frac{d}{ds} E(\alpha_s) = \frac{1}{2}\int_0^1 \frac{\partial}{\partial s}\langle \partial_t \alpha, \partial_t \alpha \rangle \, dt$$

$$= \int_0^1 \langle \nabla_{\partial_s} \partial_t \alpha, \partial_t \alpha \rangle \, dt$$

symmetry Lemma $$= \int_0^1 \langle \nabla_{\partial_t} \partial_s \alpha, \partial_t \alpha \rangle \, dt$$



I.B.P. $$= \int_0^1 -\langle \partial_s \alpha, \nabla_{\partial_t} \partial_t \alpha \rangle dt + \langle \partial_s \alpha, \partial_t \alpha \rangle \Big|_{t=0}^{t=1}$$

set $s=0$.

$$E_*(\delta\alpha) = \int_0^1 -\langle \delta\alpha, \nabla_{\partial_t} \partial_t \alpha \rangle \, dt$$

$\delta\alpha = \frac{d}{ds}\Big|_{s=0} \alpha$.

- **2nd Variation Formula:**

  - γ is a geodesic $\quad \nabla_{\dot\gamma}\dot\gamma = 0.$

  - $\alpha_{s_1,s_2}(t). \qquad s_i \in (-\varepsilon, \varepsilon).$

    $\alpha_{0,0}(t) = \gamma(t)$

  - $\dfrac{\partial^2}{\partial s_1 \partial s_2} E(\alpha_{s_1,s_2}).$

$$= \frac{\partial}{\partial s_1}\left(\frac{1}{2}\int_0^1 \frac{\partial}{\partial s_2}\langle \partial_t\alpha, \partial_t\alpha\rangle \, dt\right)$$

$$= \frac{\partial}{\partial s_1}\left(-\int_0^1 \left\langle \frac{\partial}{\partial s_2}\alpha, \nabla_{\partial_t}\partial_t\alpha\right\rangle \cdot dt\right)$$

↑ not zero ∵ (s_1, s_2) are free.

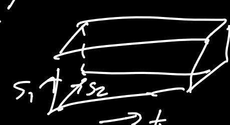
$$= -\int_0^1 \left\{\left\langle \nabla_{\partial_{s_1}}\frac{\partial}{\partial s_2}\alpha, \nabla_{\partial_t}\partial_t\alpha\right\rangle + \left\langle \frac{\partial}{\partial s_2}\alpha, \nabla_{\partial_{s_1}}\nabla_{\partial_t}\cdot\partial_t\alpha\right\rangle\right\} dt$$

$S = (-\varepsilon, \varepsilon)\times(-\varepsilon, \varepsilon)\times[0,1]$

$\alpha : S \to M.$

now, setting $(s_1, s_2) = (0,0)$, then $\nabla_{\partial_t}\partial_t\alpha = 0.$

$$= -\int_0^1 \langle \partial_{s_2}\alpha, \nabla_{\partial_{s_1}}\nabla_{\partial_t}\cdot\partial_t\alpha\rangle \, dt$$



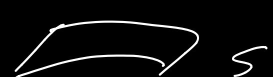
$\alpha^*(TM)$

$$\nabla_{\partial_{s_1}}\nabla_{\partial_t} - \nabla_{\partial_t}\nabla_{\partial_{s_1}} - \cancel{\nabla_{[\partial_{s_1},\partial_t]}} \to 0$$

$$= R(\partial_{s_1}, \partial_t) \quad \text{on } \alpha^*(TM).$$

$\partial_t, \partial_{s_1} \in \Gamma(S, TS)$

$\partial_t\alpha \in \Gamma(S, \alpha^*(TM)).$



$$= -\int_0^1 \left\langle \partial_{s_2}\alpha, \left(\nabla_{\partial_t}\nabla_{\partial_{s_1}} + R(\partial_{s_1},\partial_t)\right)\cdot\partial_t\alpha\right\rangle dt.$$

$E_{**}(\partial_{s_1}\alpha, \partial_{s_2}\alpha)$

$$= -\int_0^1 \left\langle \partial_{s_2}\alpha, \underbrace{\nabla_{\partial_t}\nabla_{\partial_t}}_{=\nabla^2_{\partial_t}}\cdot\partial_{s_1}\alpha + R(\partial_{s_1},\partial_t)\cdot\partial_t\alpha\right\rangle dt.$$

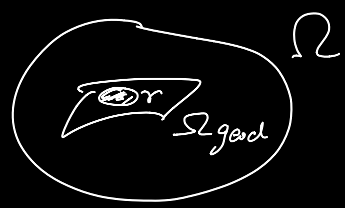
$E_{**}(\delta_1\alpha, \delta_2\alpha)$

$$= -\int_0^1 \left\langle \delta_2\alpha, \nabla^2_{\partial_t}\cdot\delta_1\alpha + R(\delta_1\alpha, \dot\gamma)\cdot\dot\gamma\right\rangle dt$$

$$E_{**}(\delta_1\alpha, \delta_2\alpha) = E_{**}(\delta_2\alpha, \delta_1\alpha).$$

Jacobi Field. : $\Omega := \{\gamma : [0,1] \to M \ , \ \text{sm}\}.$

- $\Omega_{geod} \subset \Omega$ $\qquad T\Omega_{geod} = ?$



- Fix $\gamma \in \Omega_{geod}$.
- Consider a family of geodesics.

$\alpha_s(t).$ $\qquad s \in (-\varepsilon, \varepsilon).$

- **Prop**: $\delta\alpha := \left.\frac{d}{ds}\right|_{s=0} \alpha_s(t).$ satisfy.

$$\nabla^2_{\partial_t}\delta\alpha = -R(\delta\alpha, \dot\gamma)\dot\gamma$$

Pf: $\nabla_{\partial_t}\cdot\nabla_{\partial_t}\cdot\partial_s\alpha$

$$= \nabla_{\partial_t}\nabla_{\partial_s}\partial_t\alpha = \left(R(\partial_t,\partial_s) + \nabla_{\partial_s}\nabla_{\partial_t}\right)\partial_t\alpha.$$

$$= R(\partial_t,\partial_s)\cdot\partial_t\alpha + \nabla_{\partial_s}(\underbrace{\nabla_{\partial_t}\partial_t\alpha}_{\equiv 0 \ \forall s \in (-\varepsilon,\varepsilon)})$$

$$= -R(\partial_s,\partial_t)\partial_t\alpha. \quad \blacksquare$$

- **Def**: · Let γ be a geodesic.
  · Let $J \in$ "$\Gamma(\gamma, TM|_\gamma)$". a tangent v.f. along γ. $J(t) \in T_{\gamma(t)} M$. $t \in [0,1]$.

  · J is a Jacobi field, if

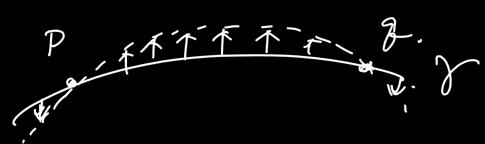
$$\nabla^2 J + R(J, \dot\gamma)\dot\gamma = 0. \quad \text{--- Jacobi Eq.}$$

- **Cor**: J is a Jacobi field, iff.

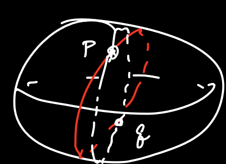
$$E_{**}(J, -) = 0.$$

- **Def**: (Conjugate points) Let γ be a geodesic.
  we say $p, q \in \gamma$ are conjugate points,
  if $\exists J$ Jacobi field, s.t. $J(p) = 0$, $J(q) = 0$.

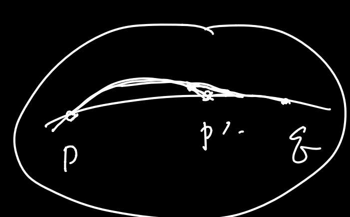
Ex

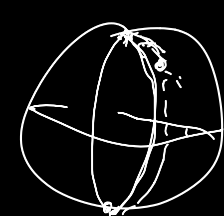


Ex: S^2 : antipodal points are conjugate points.



① If γ is a geodesic from p to q. that has point $p' \in \mathrm{int}(\gamma)$, s.t. (p, p') are conjugate
then. γ is **not** a minimizing geodesic.

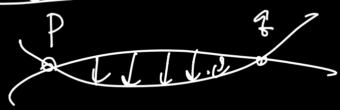




② If we have sectional curvature non-positive, then there are no conjugate pair points along any γ.

proof by contradiction

Pf :



$$\langle J, \nabla^2_{\partial t} J + R(J, \dot{\gamma})\dot{\gamma} \rangle = 0$$

$$\langle J, \nabla_t^2 J \rangle = - \underbrace{\langle R(J,\dot{\gamma})\dot{\gamma}, J \rangle}_{\text{sectional curvature}}$$

$$- \underline{\int |\nabla_t J|^2 dt} \geq 0$$

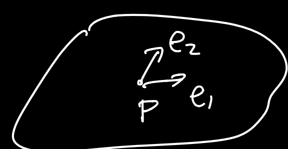
$$\Rightarrow \int |\nabla_t J|^2 dt = 0$$

$$\nabla_t J = 0.$$

$$J(p) = 0, \quad J(q) = 0.$$

$$\Rightarrow J(t) = 0 \quad \forall t$$

$$\langle R(X,Y)Z, W \rangle.$$



$e_1 \perp e_2$

$\|e_1\|^2 = \|e_2\| = 1.$

$K(e_1,e_2) := \underline{\langle R(e_1, e_2) e_2, e_1 \rangle}$, sectional curvature at pt p in the 2 plane span (e_1, e_2)