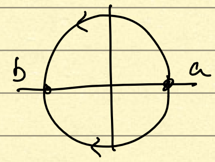
Comment on HW10:

#4. 

$A = x\,dy - y\,dx. = \Gamma_1\,dx + \Gamma_2\,dy.$ $\quad \Gamma_1 = -y,\ \Gamma_2 = x.$

$F = dA = 2\,dx \wedge dy$

$u(t)$: fiber coordinate $\qquad a = b = 1.$

parallel transport equation: $\quad \dot{u}^a(t) + \Gamma^a_{i\,b}\,\dot{\gamma}^i\,u^b(t) = 0$

$\Leftrightarrow \quad \dot{u}(t) + \underset{\sim}{\Gamma_i} \cdot \underset{\sim}{\dot{\gamma}^i} \cdot u(t) = 0$

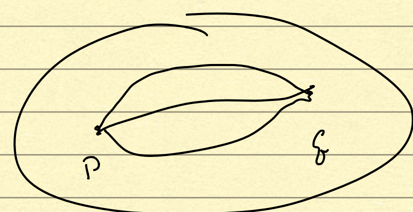
$\gamma(t) = (\cos t, \sin t) \qquad t \in (0, \pi).$

$\dot{\gamma}(t) = (-\sin t, \cos t).$

$\Gamma_i \cdot \dot{\gamma}^i(t) = -y\big|_{\gamma(t)} \cdot \dot{\gamma}^1(t) + x \cdot \dot{\gamma}^2(t) = -\sin t \cdot (-\sin t) + \cos t \cdot \cos t = 1.$

$\dot{u}(t) + u(t) = 0 \qquad \Rightarrow \qquad u(t) = e^{-t} \cdot u(0).$

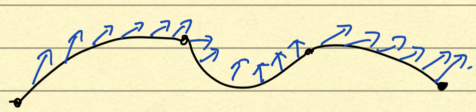
Today:
- First and Second Variation Formula.
- Jacobi field.



- (M, g), $\quad p, q \in M.$
- $\Omega_{p,q}$ = the space of piecewise smooth path from p to q.

$\gamma \in \Omega_{p,q}$: $\gamma : [0, 1] \to M.$ s.t. $\exists.\ 0 < t_1 < t_2 < \cdots < t_N < 1$, $\gamma : (t_i, t_{i+1}) \to M$ is smooth

- $T_\gamma \Omega_{p,q}$: the space of vector fields along γ. valued in TM.

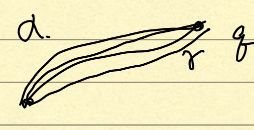
γ: 

- A variation of $\gamma \in \Omega_{p,q}$ is a continuous map:

$$\alpha : (-\varepsilon, \varepsilon)_s \times [0, 1]_t \to M.$$

$\alpha_s(t) = \alpha(s, t)$

- $\forall s \in (-\varepsilon, \varepsilon)$; $\alpha_s \in \Omega_{p,q}$, $\alpha_0 = \gamma$.
- (technical assumption): $\exists.\ 0 = t_0 < t_1 < \cdots < t_K = 1$, s.t. $\alpha|_{(-\varepsilon,\varepsilon) \times (t_i, t_{i+1})}$ is smooth.



- infinitesimal variation:

$$\delta\alpha := \frac{\partial \alpha_s}{\partial s}\Big|_{s=0}$$

· Energy function : $E : \Omega_{p,q} \to \mathbb{R}$

$$E(\gamma) = \frac{1}{2}\int_0^1 |\dot\gamma(t)|_g^2 \, dt$$

$\forall t \in (0,1).$ $\Delta_t \dot\gamma = \lim_{h\to 0^+} \left(\dot\gamma(t+h) - \dot\gamma(t-h)\right)$ ↙ (computed in certain charts, or parallel transport to $\gamma(t)$, then compare.)

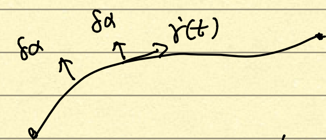
(not laplacian!)

is nonzero only at finitely many t.s.

Thm : $E : \Omega \to \mathbb{R}$, $\delta\alpha \in T_\gamma\Omega$.

$$E_*(\delta\alpha) := \frac{d}{ds}\Big|_{s=0} E(\alpha_s)$$

$$= -\sum_t \langle \delta\alpha(t), \Delta_t \dot\gamma \rangle - \int_0^1 \langle \delta\alpha, \underset{\sim\sim}{\nabla_{\frac{d}{dt}}} \dot\gamma \rangle \, dt$$

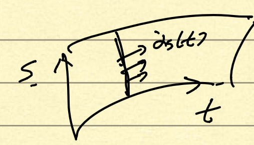


Pf: $E = \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2 \, dt$

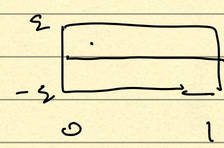
$$\frac{d}{ds}E = \frac{1}{2}\int_0^1 \frac{\partial}{\partial s}|\dot\alpha_s(t)|_g^2 \, dt$$

$\dot\alpha_s(t) = \frac{d}{dt}\alpha_s$.

$$\frac{\partial}{\partial s}|\dot\alpha_s(t)|^2 = \frac{\partial}{\partial s} g(\dot\alpha_s(t), \dot\alpha_s(t))$$



$$= g\left(\nabla_{\frac{\partial}{\partial s}}(\dot\alpha_s(t)), \dot\alpha_s(t)\right) \cdot 2.$$



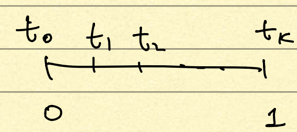
Lemma (Symmetric). (Read in Lee's Ch6).

$$\nabla_{\frac{\partial}{\partial s}}\left(\frac{\partial}{\partial t}\alpha(s,t)\right) = \nabla_{\frac{\partial}{\partial t}}\left(\frac{\partial}{\partial s}\cdot\alpha(s,t)\right).$$

this can be verified in local coordinates.

on $[-\varepsilon, \varepsilon) \times [0,1]$.

(morally speaking : $[\nabla_{\frac{\partial}{\partial t}}, \nabla_{\frac{\partial}{\partial s}}]$ but, it is acting ~~on~~ not on the tangent bundle, $\alpha^* TM$, but on the trivial bundle, functions on $C^\infty((-\varepsilon,\varepsilon)\times[0,1])$.)



→ integrate along $[0,1]$.

$$E = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \langle \nabla_{\frac{\partial}{\partial t}} \delta\alpha, \dot\gamma \rangle \cdot dt$$

integration by parts

$$\underset{\sim\sim}{\nabla_{\frac{\partial}{\partial t}}}\left(g(\delta\alpha, \dot\gamma)\right) = g\left(\nabla_{\frac{\partial}{\partial t}}(\delta\alpha), \dot\gamma\right) + g\left(\delta\alpha, \nabla_{\frac{\partial}{\partial t}}(\dot\gamma)\right)$$

$$g(\delta\alpha, \dot\gamma)\Big|_{t_{i-1}+0}^{t_i - 0} = \int_{t_{i-1}}^{t_i} g\left(\nabla_{\frac{\partial}{\partial t}}(\delta\alpha), \dot\gamma\right) + g\left(\delta\alpha, \nabla_{\frac{\partial}{\partial t}}\dot\gamma\right) \, dt$$

sum over i, get the result. contribute $E_*(\delta\alpha)$.

$$E_*(\delta\alpha) = -\sum\langle \delta\alpha, \Delta_t(\dot{\gamma})\rangle - \int_0^1 \langle \delta\alpha, \nabla_{\frac{d}{dt}}\cdot\dot{\gamma}\rangle\, dt. \quad \#.$$

$\Rightarrow$ The critical point of E are geodesics. ie γ. s.t. $\nabla_{\frac{d}{dt}}\dot{\gamma} = 0$.

Second variation formula : Hessian of E at a crit pt

2-param variation of a geodesic.

$$\alpha : \underbrace{(-\varepsilon, \varepsilon)}_{s_1} \times \underbrace{(-\varepsilon, \varepsilon)}_{s_2} \times \underbrace{[0,1]}_{t} \to M.$$

$$(s_1, s_2, t) \mapsto \alpha_{s_1, s_2}(t)$$

$\exists\ 0 < t_0 < t_1 < \cdots < t_k = 1$, s.t. on $\overbrace{(t_i, t_{i+1})}^{(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times}$, α is smooth.

$$\delta_i\alpha = \left.\frac{\partial\alpha}{\partial s_i}\right|_{(s_1,s_2)=(0,0)}.$$

$$E_{**}(\delta_1\alpha, \delta_2\alpha) := \left.\frac{\partial^2 E(\alpha_{s_1,s_2}(t))}{\partial s_1\, \partial s_2}\right|_{(0,0)}.$$

<u>Thm</u>:
$$E_{**}(\delta_1\alpha, \delta_2\alpha) = -\sum_t \langle \delta_2\alpha, \Delta_t \delta_1\alpha\rangle - \int_0^1 \langle \delta_2\alpha, \nabla^2_{\frac{\partial}{\partial t}} \delta_1\alpha - R(\dot{\gamma}, \delta_1\alpha)\cdot\dot{\gamma}\rangle\cdot dt.$$

<u>Pf</u>:
$$\frac{\partial E}{\partial s_2} = -\sum_t \left\langle \delta_2\alpha, \Delta_t \frac{\partial\alpha}{\partial t}\right\rangle - \int_0^1 \left\langle \delta_2\alpha, \nabla_{\frac{\partial}{\partial t}} \frac{\partial\alpha}{\partial t}\right\rangle dt.$$

$$\frac{\partial}{\partial s_1}\frac{\partial E}{\partial s_2} = -\sum_t \left\langle \nabla_{\frac{\partial}{\partial s_1}}(\delta_2\alpha), \Delta_t \frac{\partial\alpha}{\partial t}\right\rangle - \left\langle \delta_2\alpha, \nabla_{\frac{\partial}{\partial s_1}}\left(\Delta_t \frac{\partial\alpha}{\partial t}\right)\right\rangle$$
$$- \int_0^1 \left\langle \nabla_{\frac{\partial}{\partial s_1}} \delta_2\alpha, \nabla_{\frac{\partial}{\partial t}} \frac{\partial\alpha}{\partial t}\right\rangle + \left\langle \delta_2\alpha, \nabla_{\frac{\partial}{\partial s_1}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial\alpha}{\partial t}\right\rangle dt.$$