# Calculus of Variation:

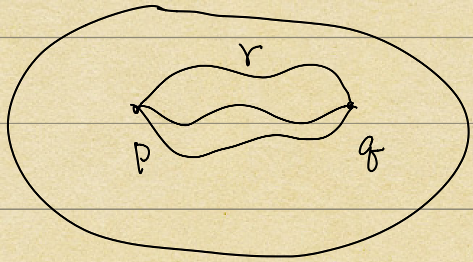
Ref 1. John Milnor : 《Morse Theory》.

2. Lee's 《Riemannian Geometry》 Ch10.

3. Nicolescu. §5.2 (follow Milnor's approach)

- Path space: Let $(M,g)$ be a sm Riem. mfd.

$p,q \in M$. $\mathcal{P} = \mathrm{Path}^{sm}(p,q) = \{\gamma : [0,1] \to M,\ \gamma(0) = p,\ \gamma(1) = q.\ \gamma \text{ sm}\}$.




morally speaking: Path space is an infinite dim mfd., we are going to do "Morse Theory" on the path space.

Crash course on Morse Theory:

$(M,g)$ Rm mfd. $f : M \to \mathbb{R}$ sm.

critical points of $f$ : $\{p \in M,\ df(p) = 0\}$

critical values of $f$ : image of crit pts.

- we say a critical pt $p$ is non-degenerate, if the Hessian of $f$ at $p$ is non-deg bilinear form.

$\mathrm{Hess}_p f : T_pM \times T_pM \to \mathbb{R}$.

in local coordinate $\mathrm{Hess}_p(f)(\partial_i, \partial_j) = \partial_i \partial_j f|_p$.

Ex: $f = x^2 - y^2$, $\mathrm{crit}(f) = \{(0,0)\}$.

$\mathrm{Hess} f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ non-degenerate.

( • around a non deg crit pt: $\exists$ coord, s.t. $f = \sum_{i=1}^{n-\mu} x_i^2 - \sum_{i=n-\mu+1}^{n} x_i^2$. )

- a bilinear form $B : V \times V \to \mathbb{R}$ is non-deg

$\Leftrightarrow$ in a trivialization $V \cong \mathbb{R}^n$, $B$ as $n \times n$ matrix is invertible.

$\Leftrightarrow$ $\forall v \in V$, $\exists w \in V$, s.t. $B(v,w) \neq 0$.

Ex: degenerate crit pt : $f: \mathbb{R}^3 \to \mathbb{R}$. $f = x \cdot y \cdot z$.

- $\underline{\text{crit}(f)} = \{x=0\} \cup \{y=0\} \cup \{z=0\} = f^{-1}(0)$.
- crit pts are not isolated.

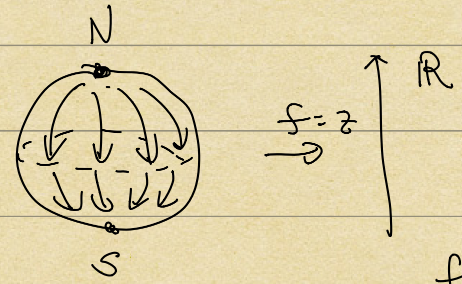
$g$ is used, to create a flow, the "(negative) gradient flow"

grad(f) := the vector field <u>dual</u> to the 1-form df

where dual is by the metric

in local chart $\quad \text{grad}(f) = \sum_{i,j} \underline{g^{ij}(x)} \cdot \underbrace{\partial_i f(x)}_{\text{components of } df} \cdot \underbrace{\partial_j}_{\text{basis of v.f.}}$

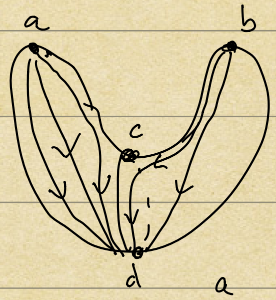
- Ex:
  $S^2$:




crit(f) = {N, S}.

flow lines of f goes between crit pts.

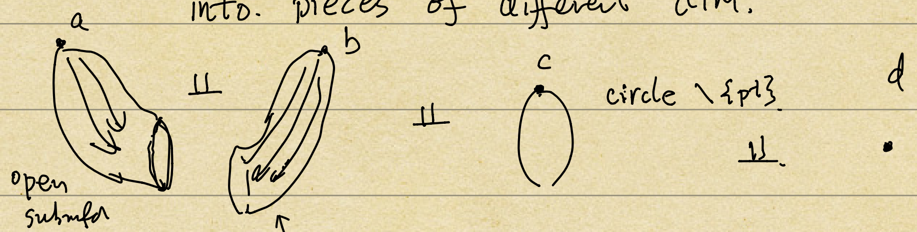
Ex:




"heart pillow" 4 crit pts.

the flow decomposes the surface into pieces of different dim.

surface =



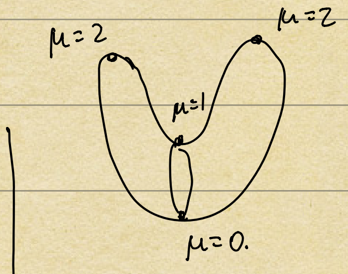

unstable mfd associated to crit pt b. w.r.t the downward grad flow.

(assume the crit pts are non-deg).

- Morse Index : $\mu_f(p)$ = # of negative eigenvalues of the bilinear form $\text{Hess}_p f$

= dim of the unstable mfd associated to crit pt p.

= "the number of unstable directions"



- "The morse function on path space $P_{p,q}$"

$\nexists$ $E(\gamma) := \int_0^1 |\dot{\gamma}(t)|^2 \, dt$ energy function.

$L(\gamma) := \int_0^1 |\dot{\gamma}(t)| dt = |\gamma|$ length functional.

$= \int_0^1 \sqrt{g(\dot{\gamma}, \dot{\gamma})} \, dt$

- it turns out studying $L(\gamma)$ can be replaced by $E(\gamma)$.
(same crit pts ---).

Thm : (1) crit "pts" of $E(\gamma)$ are geodesics connecting $p, q$.

(2). Morse index of $E(\gamma)$ at a geodesic $\gamma$
$=$ # of "conjugate points' w.r.t $p$" on $\gamma$.