Plan: (1) other type of cohomology

(2) Vector bundle. G-bundle.

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(1) Cohomology.

- in general: cochain complex, A^i: vector sp / $\mathbb{R}$

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \to \cdots \qquad d^2=0$$

$$H^i(A^*, d) = \frac{\ker(A^i \xrightarrow{d} A^{i+1})}{\mathrm{im}(A^{i-1} \xrightarrow{d} A^i)} \qquad \text{sub-quotient of } A^i.$$

- given a manifold (sm, cpt) M. we defined de Rham coh.

$$H^i_{dR}(M) := H^i(\Omega^*_{dR}(M), d_{dR}).$$

Other type of cohomology:

- Singular Cohomology:

$C^0(M, \mathbb{R})$, $C^1(M)$, ...

$C^i(M)$ = linear functions on $C_i(M)$.

- where $C_i(M) = \mathbb{R}\cdot\{\underbrace{\text{maps of } i\text{-dim simplex to } M}_{\text{basis of } C_i(M).}\}$

$C_0(M)$ has basis, map(pt, M)

$C_1(M) = \mathbb{R}\cdot \mathrm{Map}(\longrightarrow\!\bullet, M)$

- $\alpha \in C^k(M)$, assigns a number to each map from k-simplex Δ^k to M.

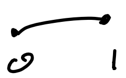
$$C^0(M) \xrightarrow{d} C^1(M) \xrightarrow{d} C^2(M) \to \cdots$$

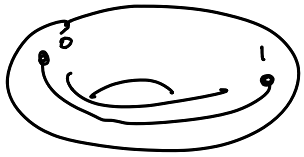
$$\varphi \in C^k(M).$$

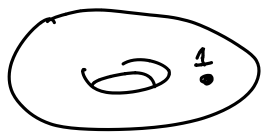
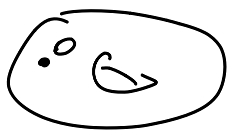
$$(d\varphi)(u) := \varphi(\delta u).$$

$u \in C_{k+1}(M)$, $\delta u \in C_k(M)$

δu is the boundary of the $(k+1)$-simplex u.

Ex: u :  $\to$ $M =$ 



$\delta u =$  $-$  $\in C_0(M)$

Δ^k k-dim simplex has k+1 vertices.

$v_0, \cdots, v_k$.

$$\partial(\Delta^k) = \sum_{i=0}^{k} (-1)^i [v_0 \cdots \hat{v}_i \cdots v_k].$$

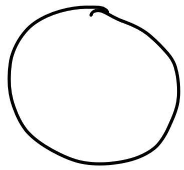
Rmk: (1) only need topological info of M,

(M. can be just a topological space).

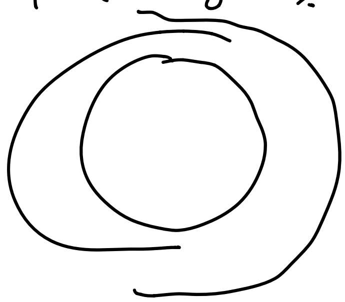
(2) good for making definition. not good for computation.

(2) Čech Cohomology.

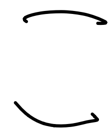
· **good cover:** we say ~~a~~ an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M is a good cover. if U_i and all ~~~~ their intersections. $U_{i_1} \cap \cdots \cap U_{i_k}$ are contractible.

Ex: $M =$  S^1.

cover #1 (not good)

U_0  U_1

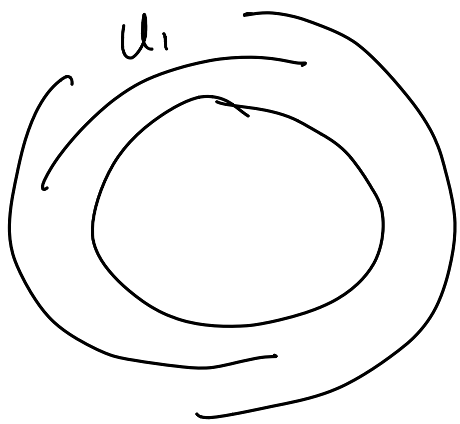
U_0, U_1 : contractible

$U_0 \cap U_1 =$  disjoint union of 2 segment

not good cover.

cover #2

U_1 U_2 U_0



$\mathcal{U} = \{U_i\}_{i \in I}$

· $\check{C}^k(M, \mathcal{U}) = \prod\limits_{\substack{J \subset I \\ |J| = k+1}} \left(\begin{array}{c} \text{constant function} \\ \text{on } U_{j_0} \cap \cdots \cap U_{j_k} \end{array} \right)$ ← good cover

$\varphi_k \in$ ... $\mathbb{R}$.

φ_k assign a number to each $U_J \neq \phi$, $|J| = k+1$,

$J = \{j_0, \cdots, j_k\}$. not just subset. ordered tuple.

assume $U_J = U_{j_0} \cap \cdots \cap U_{j_k} \neq \phi$.

· $\check{C}^k(M, \mathcal{U}) \xrightarrow{d} \check{C}^{k+1}(M, \mathcal{U})$.

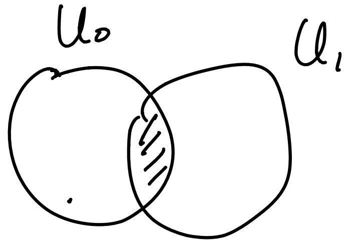
$$\varphi \longmapsto (d\varphi), \quad (d\varphi)(\underbrace{\mathcal{U}}_{\{j_0,\dots,j_{k+1}\}}) = \sum_{i=0}^{k+1} (-1)^i \cdot \varphi(\mathcal{U}_{j_0\cdots\hat{j}_i\cdots j_{k+1}}).$$

$|J| = k+2$

$\mathcal{U}_J \neq \phi$

<u>Ex</u>: $\varphi_0 \in \check{C}^0(M, \mathcal{U})$: assigns a number to each $\mathcal{U}_i$.

$$(d\varphi_0)(\mathcal{U}_{i_0 i_1}) = \varphi_0(\mathcal{U}_{i_1}) - \varphi_0(\mathcal{U}_{i_0}).$$



$\varphi_0(\mathcal{U}_0) = 3$

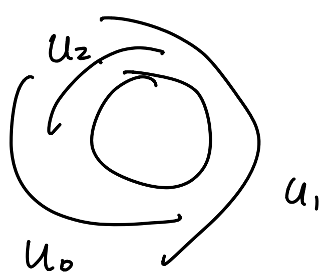
$\varphi(\mathcal{U}_1) = 5$

$(d\varphi)(\mathcal{U}_{i_0 i_1}) = 5 - 3. = 2.$

either
- fix an ordering on I, or let J be order k-tuple.

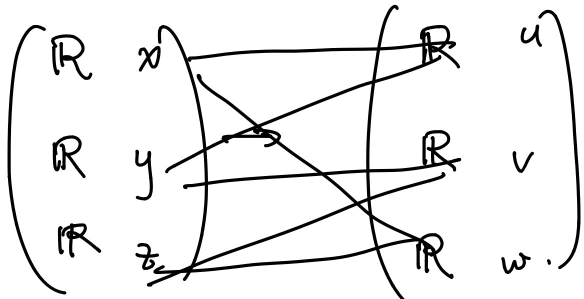
$d^2\varphi = 0$

<u>Ex</u>: cohomology of a circle S^1.



$\check{C}^0(S^1) \simeq \mathbb{R}^3.$

$\check{C}^1(S^1) \simeq \mathbb{R}^3.$



$u = x - y$

$v = y - z$

$w = x - z.$

$\ker(\check{C}^0 \to \check{C}^1) \simeq \mathbb{R}$ $\qquad$ $H^0 \simeq \mathbb{R}$

$\mathrm{coker}(\check{C}^0 \to \check{C}^1) \cong \mathbb{R}.$ $\qquad$ $H^1 \simeq (\mathbb{R})$

Rmk : (1) this can be generalized to sheaf cohomology:

Here, we use $\underline{\mathbb{R}}_M$. constant function shf. i.e.

$\underline{\mathbb{R}_M}(\mathcal{U}) = $ (constant fcns on $\mathcal{U}$.)

↑ connected open

$$(\check{C}^k(M, \mathcal{U}; \underline{\mathbb{R}}_M), d)$$

but, we can consider other sheaves. $\mathcal{F}$.

Example : M cplx mfd,

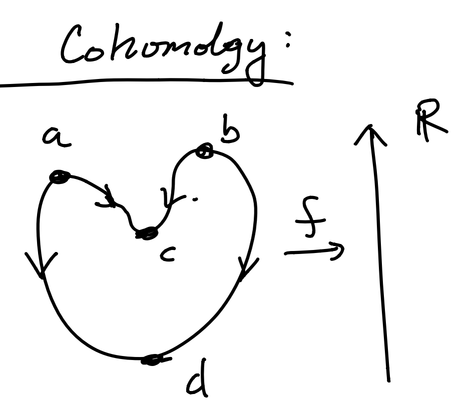
$\mathcal{F} = \mathcal{O}$ holomorphic function

$H^*(M, \mathcal{O}) = ?$

(2). One can compare Čech with de Rham cohomology, and show they are the same.

(see Nicolascu)

(3) Morse Cohomology: $(M, \overset{\text{choose metric}}{g}, \underset{\text{choose a Morse function}}{f})$.

Ex : $S^1 =$ 

$\mathrm{grad}_g f$: gradient vector field of f.

$$C^k_{Mor}(M, \underline{g}, \underline{f}) = \bigoplus_{\substack{p \in Crit(f) \\ \mu(p) = k}} \mathbb{R} \cdot \langle p \rangle$$

auxiliary choices,

$\begin{cases} \text{index of } a, b = 1 \\ \text{index of } c, d = 0. \end{cases}$

index at p = # of negative eigenvalues of $\mathrm{Hess}(f)|_p$

- $C^k_{Mor} \xrightarrow{d} C^{k+1}_{mor}.$

$$\langle p \rangle \longmapsto \sum_{\substack{q \in \mathrm{Crit}(f) \\ \mathrm{index}(q) = k+1}} \underline{\underline{C_{pq}}} \langle q \rangle$$

$\uparrow \mathbb{Z}.$

C_{pq} signed count the gradient flow lines from q to p.

Ex:

$$C^0_{Mor} = \mathbb{R} \cdot \langle c \rangle, \langle d \rangle, \simeq \mathbb{R}^2$$

$$d \downarrow$$

$$C^1_{mor} : \mathbb{R} \langle a \rangle \langle b \rangle. \simeq \mathbb{R}^2.$$

$$(x, y) \mapsto (u, v)$$

$$u = x + y$$

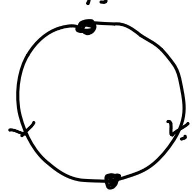
$$v = x + y.$$

$\ker(d) \simeq \mathbb{R}$

$\mathrm{im}(d) \simeq \mathbb{R}.$

in a different morse fcn on S^1.

A



B

$\mathbb{R} \xrightarrow{0} \mathbb{R}$

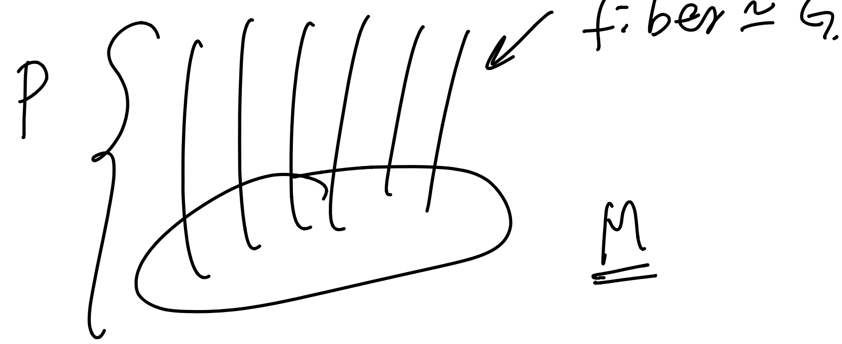
- $\underline{\mathbb{R}}_M \to \Omega^0(\cdot) \to \Omega^1 \to \Omega^2 \cdots$ (de Rham resolution of constant shf.)

$\uparrow$ shf

- $\underline{\mathbb{R}}_M \to \bigoplus_{i \in I} \underline{\mathbb{R}}_{U_i}(-) \longrightarrow \bigoplus_{i_0, i_1} \underline{\mathbb{R}}_{U_{i_0 i_1}}(-) \to \cdots$ (Čech Resolution)

(they are all different resol'n. of the same sheaf $\underline{\mathbb{R}}_M$).

- Principle G-bundle



- characteristic class of P are cohomology classes on M

- Chern - ~~The~~ Weil Theory:
  - making additional choice of connection & curvature on $P \to M$.
  - then for each "invariant polynomial" on $\varphi : \mathfrak{g} \to \mathbb{R}$

$$Ad : G \curvearrowright \mathfrak{g}.$$

$$\varphi(F) \in \Omega^*(M).$$

F: 2-form on M, valued in $Ad(P)$

vector bundle over M

fiber $\simeq \mathfrak{g}$.

- need to show
  1. $\varphi(F)$ is closed
  2. $[\varphi(F)]$ does not dep on A the choice of connection on P.

$[\varphi(F)] \in H^*(M)$