• (M, g) Riem mfd.

$S \subset M$ submfd. $g|_S$, induced metric.

∇^M : Levi-Cevita connection on TM.

∇^S : - - - - on TS.

• <u>Claim</u>: X, Y vector field on S, $p \in S$.

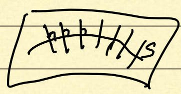
• $(\nabla^M_X Y)_p = (\nabla^M_X Y)_p^{\parallel} + (\nabla^M_X Y)_p^{\perp}$

$(\nabla^M_X Y)^{\parallel}_p = (\nabla^S_X Y)_p$

$(\nabla^M_X Y)^{\perp}_p =: N(X, Y)$. second fundamental form.

$N : \mathrm{Vect}(S) \times \mathrm{Vect}(S) \longrightarrow \Gamma(S, NS)$

NS : normal bundle of S in M



$(NS)_p = \{ v \in T_pM \mid v \perp_g w, \forall w \in T_pS \}$

$N(X, Y) = N(Y, X)$.
(from [Ni])

<u>Thm</u>: Let X, Y, Z, T be v.f. on S.

$\langle R^M(X,Y)Z, T \rangle$

$= \langle R^S(X,Y)Z, T\rangle + \langle N(X,Z), N(Y,T)\rangle$

$- \langle N(X,T), N(Y,Z)\rangle$.

[recall: Riemman curvature R is a 2-form valued in $\mathrm{End}(TM)$.]

<u>Rmk</u>: (1) (M, g) is a flat metric.

$R^M = 0$

$\therefore \; 0 = \langle R^S(X,Y)Z, T\rangle + \langle N, N\rangle - \langle N, N\rangle$

(2) S is a hypersurface; then normal bundle NS is a line bundle.

(3) $M = (\mathbb{R}^3, g_{std})$. S = surface.

(4.2.12) [Ni].

$$\langle R^S(X,Y)Z, T\rangle = \begin{vmatrix} N_n(X,T) & N_n(X,Z) \\ N_n(Y,T) & N_n(Y,Z) \end{vmatrix}.$$

<u>Ex</u>: (HW" Prob 1).

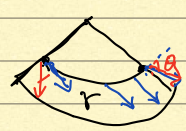
Q: parallel transport along a circle γ.

Here is an intuitive way to parallel transport.

① construct a cone that is tangent to the S^2 along γ.

② $P_\gamma^{\text{cone}} \overset{!}{=} P_\gamma^{\text{sphere}}$.

③ P_γ^{cone}



$TS^2|_\gamma = T\,\mathrm{Cone}|_\gamma \ni X$.

$\nabla^{\text{cone}}_{\dot\gamma}(X) \overset{!}{=} \nabla^{S^2}_{\dot\gamma}(X)$. ✓

both $= \left[\nabla^{\mathbb{R}^3}_{\dot\gamma}(X)\right]^{\parallel}$ ✓

Gauss - Bonnet Thm:

• Classical statement:

• (S, g) smooth compact surface with Riemn metric g.

• Scalar Curvature:

• <u>Riemman Tensor</u> $R_{ij}{}^{\alpha}{}_{\beta}$. $\approx$ 2nd derivative on g_{ij}

• <u>Ricci Tensor</u> $R_i{}^{\alpha} \overset{(\pm)}{=} R_{ij}{}^{\alpha}{}_{\beta}\, g^{j\beta}$.

• <u>Scalar curvature</u> $R = R_i{}^{i}$

$= C \cdot \chi(S)$

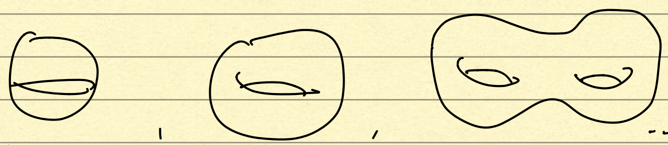
• $\int_S R \cdot d\mathrm{Vol}_g$ = const, indep of choice of g.

(1)

- Ex: · S^2 unit sphere.
  - scalar curvature $R = 1$.
  - volume $(S^2) = 4\pi$.

$\therefore \int_{S^2} R \cdot dvol_g = 4\pi$.

· $\chi(S^2) = 2 - 2g(S) = 2$. ($g$ = genus).



$g=0$, $g=1$, $g=2$. ..

$\chi = 2-2=0$

$$\int R \cdot dVol_g = 2\pi \cdot \chi(S).$$

$$4\pi = 2\pi \cdot 2.$$

(2) $S = T^2$, with flat metric.
$= \mathbb{R}^2 / \mathbb{Z}^2$

$$LHS = \int 0 \cdot dVol = 0$$

$$RHS = 2\pi \cdot \chi(T^2) = 2\pi(2-2) = 0.$$

$dVol_g$ is a top-degree diff form.

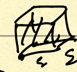
in local coord $x^1, \cdots, x^n$. $g = g_{ij}\, dx^i dx^j$

$$dVol_g = \sqrt{g}\; dx^1 \wedge \cdots \wedge dx^n.$$
$$g = \det(g_{ij}).$$

[choosing an orientation on M.
assume $dx^1 \wedge \cdots \wedge dx^n$ agree with the orientation.]

[why this formula:

volume of little cube of size ε.



$(x_1, \cdots, x_n)$

- approx take g_{ij} to be const.
- model case: $\mathbb{R}^n$, $x_1, \cdots, x_n$.
  inner product: g_{ij} (constant).
  $g_{ij} = \langle \partial_{x_i}, \partial_{x_j} \rangle$.

$\exists$ ONB, $y_1, \cdots, y_n$. $g(\partial_{y_i}, \partial_{y_j}) = \delta_{ij}$

volume form $= dy_1 \wedge \cdots \wedge dy_n$

$$dy_1 \wedge \cdots \wedge dy_n = (?) \cdot dx_1 \wedge \cdots \wedge dx_n$$

$$(?) = \underline{\det(A)}. \qquad A_{ij} = \frac{\partial y_i}{\partial x_j}$$

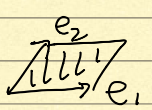
Jacobian.

$$= \sqrt{\det(G_{ij})}. \qquad G_{ij} = g(\partial_{x_i}, \partial_{x_j})$$

$$G_{ij} = g\left(\left(\sum \frac{\partial y_a}{\partial x_i}\right) \cdot \partial_{y_a},\ \left(\sum \frac{\partial y_b}{\partial x_j}\right) \frac{\partial}{\partial y_b}\right)$$
$$= A^a{}_i\, A^b{}_j\, \delta_{ab}$$

$$G = A \cdot A^T \Rightarrow \det(G) = \det(A)^2.$$

· $\mathbb{R}^2$.



$$vol(\square) = \sqrt{|e_1|^2 + |e_2|^2 - 2\langle e_1, e_2 \rangle}$$
$$= \sqrt{\det(g(e_i, e_j))}.$$