Ch 21 Quotient manifold.

(see Appendix A, Quotient space and Quotient maps.)

- G : group. M: sets.

we can talk about G orbit

$$G\cdot p = \{ g\cdot p \mid g \in G\} \subset M.$$

the orbit space M/G is the collection of orbits. So far, it is just a set.

- $\pi : M \to M/G$. $p \mapsto G\cdot p$

- If M is a topological space, we can equip the M/G a quotient topology:

a subset $U \subset M/G$ is open iff $\pi^{-1}(U)$ is open. (by construction: π is continuous).

- Goal: Find conditions on G and M, s.t. M/G has a smooth mfd structure.

Obvious conditions: · M, G sm mfd. G: Lie gp.

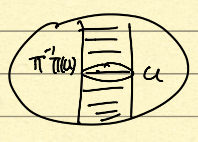
- $G \circlearrowright M$ smoothly.

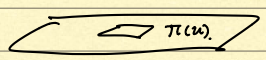
(not enough!).

Claim: if we further impose $G \circlearrowright M$ freely and properly.

Lemma 21.1: if G is a topological group. M is a top. mfd. $G \circlearrowright M$ continuously, then $\pi : M \to M/G$. is an open map. [i.e. π(open) is open).

Pf: given $U \subset M$ open, $\pi(U) = \pi(\pi^{-1}\pi(U))$





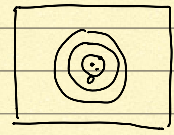
to prove that $\pi(U)$ is open $\Leftrightarrow$ $\pi^{-1}(\pi(U))$ is open.

$$\pi^{-1}\pi(U) = \bigcup_{g\in G} g\cdot U$$ arbitrary union of open sets is open.

$\downarrow$ is also open. #

Ex: ① $G \circlearrowright M$ trivially, $g\cdot p = p$. of course $M/G = M$ is smooth.

②. $S^1 \circlearrowright \mathbb{C}$



the orbits are labelled by radius. r.

$r \in [0,\infty)$.

$\mathbb{C}/S^1 = [0,\infty)$ not a manifold.

(3) $SO(n) \circlearrowright \mathbb{R}^n$ by rotation.

again the orbits are concentric spheres S^{n-1}, we can label the orbits by its radius.

$\mathbb{R}^n / SO(n) \simeq [0,\infty)$. not a mfd.

(4) $S^1 \circlearrowright S^3$ Hopf fibration.

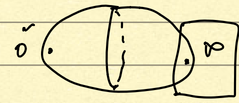
$S^3/S^1 \simeq S^2$ (or $\mathbb{CP}^1$).

$S^1 \circlearrowright S^3$ with other weights

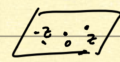
$$e^{i\theta}\cdot(z,w) = (e^{i\theta\cdot a}z, e^{i\theta\cdot b}w).$$ $S^1\cdot(1,0)$ $\boxed{S^1\cdot(0,1)}$

$a,b \in \mathbb{Z}$.

For example: $(a,b) = (1,2)$., we will see the quotient is not a manifold:



is like quotient of $\mathbb{C}/(\mathbb{Z}/2)$. = not a mfd.



just $\mathbb{R}/(\mathbb{Z}/2) \simeq [0,\infty)$.

(5) $GL(n,\mathbb{R}^n) \circlearrowright \mathbb{R}^n$.

is $p \in \mathbb{R}^n$, $GL(n,\mathbb{R})\cdot p = \mathbb{R}^n \setminus \{0\}$.

$$\mathbb{R}^n / GL(n,\mathbb{R}^n) = \{ \overset{a}{\overset{\shortparallel}{[0]}}, \overset{b}{\overset{\shortparallel}{[\mathbb{R}^n\setminus\{0\}]}} \}.$$

✱ the quotient is not Hausdorff.

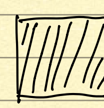
open sets here: $\{b\}$, $\{a,b\}$, ϕ.

(action is not free, causing trouble at 0).

Ex: (21.3) Consider $\mathbb{R}$ acts on T^2.

$$t\cdot(e^{i\theta}, e^{i\phi}) = (e^{i(\theta+t)}, e^{i(\phi+ct)})$$

for $c \in \mathbb{R}\setminus\mathbb{Q}$.



The open sets invariant under group actions. (precisely the preimage of open sets of $T^2/\mathbb{R}$).

are just T^2, ϕ. So $T^2/\mathbb{R}$ has trivial topology.

trouble:

(orbit gets too close to itself.)

· **Definition**: (1) $G \circlearrowright M$ freely if $\forall\, p \in M$.
$G_p := \{ g \in G \mid g\cdot p = p \} = \{e\}$.

(2) $G \circlearrowright M$ properly, if. the map
$\Theta: G \times M \to M \times M$.
$(g, p) \mapsto (p, g\cdot p)$.
is a proper map.

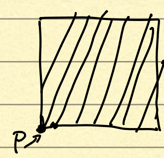
proper map: $f: M \to N$. is proper if f^{-1}(compact) is compact.

$\overset{①}{\Rightarrow}$ given $(p, q) \in M \times M$.
$\Theta^{-1}((p,q))$ should be a compact subset in G.

Prop 21.5: (Characterization of Proper Action)
Let M be a (topological) mfd.
$G \circlearrowright M$ continuously. TFAE.
(1) G acts properly
(2). If (p_i) is a seq in M.
(g_i) is a seq in G.
if both (p_i) and $(g_i p_i)$ converges in M
then (g_i) converges.

(3). For every compact subset $K \subset M$,
(the approx. stabilizer).
$$G_K = \{ g \in G \mid K \cap g\cdot K \neq \phi \}.$$
is compact. (G_K is not a group)

Apply (2) to check that, $\mathbb{R} \circlearrowright T^2$ (irrationally) is not



· $p_i = p$ const.
$g_i \in \mathbb{R}$. $\to \infty$
$(-R, R) \in \mathbb{R}$.
$(-R, R)\cdot p. \subset \mathbb{R}\cdot p$.

· $g_i p \to p$

Thm: $G \circlearrowright M$ Lie gp acts smoothly. freely. properly on a sm mfd M, then M/G can be equipped unique smooth structure, s.t.
$$\pi: M \to M/G.$$
smooth submersion.