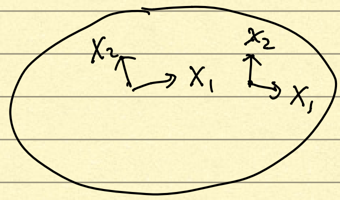
Cartan's Moving Frame:

- Let (M^n, g) be a Riemannian mfd.
- Let $(X_\alpha)_{\alpha=1,\cdots,n}$ be a local <u>orthonormal</u> frame of TM.



$$\langle X_\alpha(p), X_\beta(p)\rangle_g = \delta_{\alpha\beta} \qquad \forall p \in U.$$

(• use Gram-Schmidt for existence.)

- $\{\theta^\alpha\}_{\alpha=1}^n$, dual frame on T^*M.

  ↑ collections of 1-form.

- $$\nabla X_\alpha = \sum_{\beta=1}^n \underset{\text{1-form}}{\omega_\alpha^\beta} \otimes \underset{\text{section of } TM}{X_\beta} \qquad (*)$$

This defines a collection of 1-forms ω_α^β.

[• eg. given a vector field Y,

$$\nabla_Y X_\alpha = \sum_{\beta=1}^n \omega_\alpha^\beta(Y)\cdot X_\beta.$$

Prop:
- antisymmetric.

$$\omega_\alpha^\beta = -\omega_\beta^\alpha$$

- dual to $(*)$.

$$d\theta^\alpha = \theta^\beta \wedge \omega_\beta^\alpha \quad \left(\begin{matrix}\text{summing} \\ \text{over } \beta \\ \text{implicitly}\end{matrix}\right)$$

Pf (1): we consider inner product with X_γ. $\forall Y \in \mathfrak{X}(M)$.

$$\langle \nabla_Y X_\alpha, X_\gamma\rangle + \langle X_\alpha, \nabla_Y X_\gamma\rangle.$$

$$= \nabla_Y(\langle X_\alpha, X_\gamma\rangle) \qquad (\because \nabla \text{ preserves } g)$$

$$= 0$$

$$\Rightarrow 0 = \langle \omega_\alpha^\beta(Y) X_\beta, X_\gamma\rangle + \langle X_\alpha, \omega_\gamma^\sigma(Y) X_\sigma\rangle$$

$$= \omega_\alpha^\beta(Y)\cdot \delta_{\beta\gamma} + \omega_\gamma^\sigma(Y)\,\delta_{\alpha\sigma}$$

$$= \omega_\alpha^\gamma(Y) + \omega_\gamma^\alpha(Y) = 0$$

$$\Rightarrow \quad \omega_\alpha^\gamma = -\omega_\gamma^\alpha \qquad \forall \alpha, \gamma.$$

(2). To verify, we insert basis sections X_γ, X_σ.

$$d\theta^\alpha(X_\gamma, X_\sigma) = X_\gamma(\underbrace{\theta^\alpha(X_\sigma)}_{\text{const}})$$

$$- X_\sigma(\overbrace{\theta^\alpha(X_\gamma)}^{\text{const}}) - \theta^\alpha([X_\gamma, X_\sigma]).$$

$$= -\theta^\alpha([X_\gamma, X_\sigma]).$$

$$[X_\gamma, X_\sigma] = \nabla_{X_\gamma} X_\sigma - \nabla_{X_\sigma} X_\gamma.$$

given this.

$$-\theta^\alpha([X_\gamma, X_\sigma]) = -\theta^\alpha(\nabla_{X_\gamma} X_\sigma - \nabla_{X_\sigma} X_\gamma)$$

$$= -\theta^\alpha(\omega_\sigma^\beta(X_\gamma)\cdot X_\beta - \omega_\gamma^\beta(X_\sigma)\cdot X_\beta)$$

$$= -\omega_\sigma^\alpha(X_\gamma) + \omega_\gamma^\alpha(X_\sigma) = \text{LHS}.$$

$$\text{RHS} = (\theta^\beta \wedge \omega_\beta^\alpha)(X_\gamma, X_\sigma)$$

$$= \theta^\beta(X_\gamma)\cdot \omega_\beta^\alpha(X_\sigma) - \theta^\beta(X_\sigma)\,\omega_\beta^\alpha(X_\gamma)$$

$$= \omega_\gamma^\alpha(X_\sigma) - \omega_\sigma^\alpha(X_\gamma) = \text{RHS}.$$

LHS = RHS ✓.

- These $\omega^\alpha{}_\beta$ are matrix-valued 1-forms, viewing α, β as row/column indices.

- For $\{X_\alpha\}$ local frame, over $\mathcal{U}$ of TM, we have a local connection d on $TM|_{\mathcal{U}}$.

so. $\nabla = d + A$. "$(A = \omega)$".

$$\nabla X_\alpha = dX_\alpha + A \cdot (X_\alpha)$$
$$= 0 + A \cdot (X_\alpha)$$
$$\doteq \sum_\beta \omega_\alpha{}^\beta \cdot X_\beta$$

- curvature.

"$R = d\omega + \omega \wedge \omega$." equation of matrix-valued 2-form.

i.e.

$$R^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta$$

as equations of 2-form.

- Read [Ni] for the example of

$$g = \frac{dx^2 + dy^2}{y^2}.$$

and see if Problem 2 and 3 can be done this way.

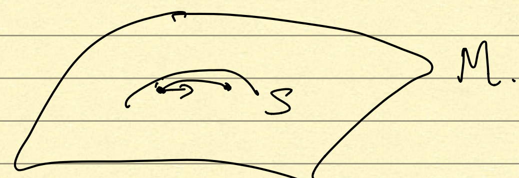
$$g_{ij} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

- Submanifold:

(M, g) Riem mfd.

$S \subset M$ nice submanifold.

$g|_S$ is a metric on S.



let γ be a path on S.

Say. $X_0 \in T_{\gamma(0)} S \subset T_{\gamma(0)} M$.

How to do parallel transport along γ of X_0?

- let $\tilde{\nabla}$ be Lev-Civita on M

∇ be Levi-Civita on S

X, Y be v.f. on S.

let $\tilde{X}, \tilde{Y}$ be some extension of X, Y to tubular nbhd around S.

$$(\tilde{\nabla}_X Y)|_p := (\tilde{\nabla}_{\tilde{X}} \tilde{Y})|_p$$ indep of extension

$$(\tilde{\nabla}_X Y)|_p = (\tilde{\nabla}_X Y)^{\parallel}_p + (\tilde{\nabla}_X Y)^{\perp}_p.$$

$$(\tilde{\nabla}_X Y)^{\parallel}_p \in T_p S \subset T_p M$$

$$(\tilde{\nabla}_X Y)^{\perp} \in (T_p S)^{\perp} \subset T_p M.$$

$$T_p M = T_p S \oplus (T_p S)^{\perp}.$$

<u>Claim</u>:

(1) $(\tilde{\nabla}_X Y)^{\parallel} = \nabla_X Y$.

cov on M then projection to TS.

cov on S

(2) $N(X, Y) = (\tilde{\nabla}_X Y)^{\perp}$

then it is symmetric.

$N(X, Y) \in NS = (TS)^{\perp}$ is call "second fundamental form."