Next : principal G-bundle , Quotient manifold.

- Ch. 21. of Lee's book.
- Kobayashi – Nomizu. Ch 1. Section 4.5. fibre bundle. (Lie group)
- Gallot - Hulin - Lafontaine : §1.E. Homogeneous space.

Last time : Maurer - Cartan form.

- 1-form on G, valued in the Lie algebra. $\mathrm{Lie}(G) = \mathfrak{g}$
- $\omega : TG \longrightarrow \mathfrak{g}$
  $(g, v) \longmapsto (L_{g^{-1}})_*(v) \in T_eG = \mathfrak{g}.$
- we showed last time, if one is given 2 left invariant v.f. X, Y on G, $\forall g \in G$.

$$d\omega(X, Y) + [\underbrace{\omega(X)}_{\mathfrak{g}}, \underbrace{\omega(Y)}_{\mathfrak{g}}] = 0$$

This is true for any vector fields X, Y.

$$(X, Y) \longmapsto F(X, Y) := d\omega(X, Y) + [\omega(X), \omega(Y)]$$

is $C^\infty(G)$ - linear in X and Y,

i.e. for any smooth function $f, g \in C^\infty(G)$,

$$F(fX, gY) = f \cdot g \cdot F(X, Y).$$

this is because $d\omega(-,-)$, $\omega(-)$ are $C^\infty(G)$ linear.

contrast with Lie derivative :

$\mathcal{L}_X(Y)$ is only $\mathbb{R}$- linear in X and Y. , not $C^\infty(M)$ - linear.

$$\mathcal{L}_{fX}(Y) = [fX, Y] = f[X, Y] + X[f, Y] = f[X, Y] - X\,Y(f).$$

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Principal G- bundle

Def : Let M be a manifold. G be a Lie group. A principal (fibre) bundle over M with group G. consist of a mfd P, and action of G on P on the right, satisfying the following condition :

(1) G acts freely on P
$P \times G \to P \qquad (u, a) \mapsto ua = R_a u \in P.$

(2) M is the quotient space of P by this action. ($M = P/G$), and the canonical projection : $\pi : P \to M$ is smooth.

(3) P is locally trivial.

recall our discussion on fiber bundle. $\forall x \in M$. $\exists\, \mathcal{U} \ni x$ open nbhd, s.t.
$\Phi : \pi^{-1}(\mathcal{U}) \xrightarrow{\sim} \mathcal{U} \times G$.
s.t. Φ is compactible with the right-action of G.

Ex : ① trivial G- bundle :
$P = M \times G \circlearrowleft G$.

② a non-trivial example?

$\{e^{i\theta}\} = S^1 \circlearrowright S^3 \subset \mathbb{C}^2.$
(S^1: abelian group)

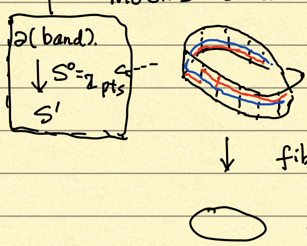
$$e^{i\theta} \cdot (z, w) = (e^{i\theta} \cdot z, e^{i\theta} \cdot w).$$

$$\nexists\, \sigma \quad \begin{array}{c} S^3 \\ \downarrow S^1 \\ \mathbb{CP}^1 \end{array}$$

Q: why is this a nontrivial S^1- bundle ?

claim: one cannot have any global section

some baby example of a non-trivial bundle :
mobius band :



↓ fiber = $\mathbb{R}$

S^1.

: zero section

: an attempt to draw a non zero section. in the bundle.

If we want to trivialize this vector bundle. we need to pick a non- zero point above every pt $p \in S^1$. So, this is non-trivial vector bundle over S^1.

Ex: Let G be a Lie group.
$H \subset G$ be a closed subgroup $\left(\begin{array}{l} H \subset G \\ \text{embedd} \\ \text{sub mfd} \end{array}\right)$

- $$\begin{array}{c} G \circlearrowleft H \\ \downarrow \text{ fiber is } H. \\ G/H \end{array}$$

G/H = space of left coset.

- $G = SU(2)$ special unitary group

$H = U(1) \subset SU(2)$

$$\left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\}. \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2). \cong S^3.$$

$$\begin{array}{c} G \\ \downarrow H \\ G/H \end{array} \qquad \bar{M}^t = M^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

$$\begin{cases} M^\dagger M = Id. \; ---- \\ \det(M) = 1 \end{cases}$$

- In general: $G = SU(n)$

$H = U(1)^{n-1}$ $\begin{pmatrix} e^{i\theta_1} & & & & 0 \\ & e^{i\theta_2} & & & \\ & & \ddots & & \\ & & & e^{i\theta_{n-1}} & \\ 0 & & & & e^{-i(\theta_1 + \cdots + \theta_{n-1})} \end{pmatrix}$

$$\begin{array}{c} G \\ \downarrow \\ G/H \end{array} = \text{flag manifold of } G.$$

Def: A homogeneous space of a Lie group G. is a manifold. M, where G acts smoothly and transitively on the left.

Ex: G/H as in the above examples.

Another construction: $\boxed{\text{associated bundle:}}$

- assume $G \circlearrowright F$ F sm mfd.

action on the left.

$P \times F = \{(u, f)\}$

$\sim$ on $P \times F$

$(ug, g^{-1}f) \sim (u, f)$.

$$P \times_G F = P \times F / \sim.$$

$$\begin{array}{c} P \times_G F \\ \downarrow F \\ M. \end{array} \qquad \begin{array}{c} P \\ \downarrow G \\ M \end{array} \qquad \begin{array}{c} P \times_G F \\ \downarrow F. \\ M \end{array}$$