Plan: 3 weeks

1.5 week $\Big[$ • Riemannian geometry

rest $\Bigg[$ • De Rham cohomology.
• Characteristic classes of vector bundle.

Covered So far:

1. general v.b. connection, curvature,

2. TM : • Levi-Cevita conn.
• geodesic.
• Riemannian curvature : Rm

• Curvature: Rm, Rc, Sc, S (Ricci, sectional, scalar)
• Comparison theorems.
"How curvature can constrain topology."

Lee «Riemann Geometry» last ~~see~~ chapter

• Geodesic. Calculus of Variation,
Jacobi field.
Milnor. «Morse Theory».

Today: • exponential maps
• Riemann normal coordinates.

• Recall that: • given a start point p
• —— starting velocity $v \in T_pM$.

we can build a geodesic:

$\gamma : (-\varepsilon, \varepsilon) \to M.$

$\gamma(0) = p, \quad \dot\gamma(0) = v.$

• Consider $D_pM = \{ v \in T_pM \mid \|v\|_g \le 1 \} \subseteq T_pM$
• we can have a map

• $\Phi : D_pM \times (-\varepsilon, \varepsilon) \to M.$

$(v, t) \mapsto \gamma_v(t)$

where γ_v is the geodesic with init cond $\dot\gamma_v(0) = v, \ \gamma_v(0) = p.$

there is a redundancy:
flow at initial velocity $2v$, with time $t/2$ is equivalent to flow — v, time t.

So, we might as well consider the flow time $t=1$, and various initial velocity.

$$\mathrm{Exp} : D_p^\varepsilon M \to M \qquad v \mapsto \gamma_v(1).$$
$$D_p^\varepsilon M = \{ v \in T_pM \mid \|v\| < \varepsilon \}$$

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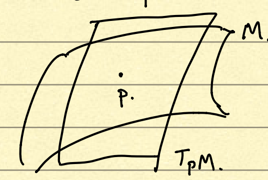
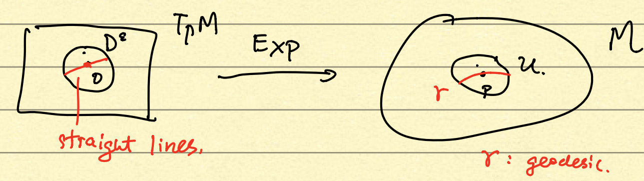
normal coordinate:
• take ONB of T_pM. wrt g.
$\{e_1, \cdots, e_n\}$ on T_pM
(living only over p).

• then we have (dual) coordinates
$\xi_1, \cdots, \xi_n$ on T_pM.
i.e. $v \in T_pM, \quad v = \xi_1 \cdot e_1 + \cdots + \xi_n \cdot e_n.$

• let $\mathcal{U} \ni p$ nbhd, $\mathcal{U} = \mathrm{Image}(\mathrm{Exp})$.



Exp is a diffeo, between small nbhd of 0 in T_pM, to a small nbhd of p in M.



RNC

• Def: normal coord on $\mathcal{U}$ associated to the exp map and $\xi_1, \cdots, \xi_n$ coord on T_pM is
$x_i = \xi_i \circ \mathrm{Exp}_p^{-1}$

• In these Riemann normal coordinates (RNC), the metric tensor g can be written nicely.

$$g = \sum_{i,j=1}^{n} g_{ij}(x) \cdot dx^i \otimes dx^j$$

$$g_{ij}(0) = \delta_{ij}, \quad (\partial_k g_{ij})(0) \overset{*}{=} 0$$

corresp to point p (proof in [Ni].)

(but $\partial_k \partial_l g_{ij}(0)$ cannot be forced to be zero)
why? The curvature maybe nonzero.

•

• $F_\nabla = \nabla^2. \qquad F_\nabla(u) = \nabla(\nabla u).$

• in local coordinates: 1-form valued in End(E).

$$\nabla = d + A, \qquad A = A^\alpha_{i\beta}\, dx^i \otimes \underset{E}{e_\alpha} \otimes \underset{E^*}{\delta^\beta}$$

$$F = \sum_{i<j} F_{ij} \, dx^i \wedge dx^j = \frac{1}{2} \sum_{i,j} F_{ij} \, dx^i \wedge dx^j$$

$$F_{ij} = F_{ij}{}^{\alpha}{}_{\beta} \; e_\alpha \otimes \delta^{\beta}.$$

$$F_{ij}{}^{\alpha}{}_{\beta} = \partial_i A_j{}^{\alpha}{}_{\beta} - \partial_j A_i{}^{\alpha}{}_{\beta} + \underbrace{\left([A_i, A_j]\right)^{\alpha}{}_{\beta}}_{A_i{}^{\alpha}{}_{\gamma} A_j{}^{\gamma}{}_{\beta} - A_j{}^{\alpha}{}_{\gamma} A_i{}^{\gamma}{}_{\beta}}$$

The above is for general vector bundle.

For $E = TM$, choose $e_\alpha = \frac{\partial}{\partial x^\alpha}$, $\delta^\beta = dx^\beta$.
$i,j \in \{1, \cdots, n\}$, $\alpha, \beta \in \{1, \cdots, n\}$.

- $R_{ij}{}^{\alpha}{}_{\beta} = \partial_i \Gamma_j{}^{\alpha}{}_{\beta} - \partial_j \Gamma_i{}^{\alpha}{}_{\beta} + \left([\Gamma_i, \Gamma_j]\right)^{\alpha}{}_{\beta}$.
  involves 2nd order derivatives of g_{ij}.
- Recall for Levi-Civita connection
  - $\Gamma_i{}^{\alpha}{}_{\beta} = \frac{1}{2} g^{\alpha\gamma} \left(-g_{i\beta,\gamma} + g_{i\gamma,\beta} + g_{\beta\gamma,i} \right)$
    involves 1st order derivative of g_{ij}
- If $R_{ij}{}^{\alpha}{}_{\beta}(p) \neq 0$, then there is no choice of coordinates, that can make
  $\partial_i \partial_j g_{kl}(p) = 0$ $\forall$ all indices i, j, k, l.

In RNC, $\because$ $\Gamma_{ik}^{j}(0) = 0$

$\therefore$ $R_{ij}{}^{\alpha}{}_{\beta}(0) = (\partial_i \Gamma_j{}^{\alpha}{}_{\beta})(0) - (\partial_j \Gamma_i{}^{\alpha}{}_{\beta})(0)$.