1. About $O(n,\mathbb{R})$ and $SO(n,\mathbb{R})$

$O(n,\mathbb{R}) \subset GL(n,\mathbb{R})$

$\| \{A \mid A^T \cdot A = Id_n\}$.

- $A, B \in O(n,\mathbb{R})$

$\underbrace{A^T A = I}$, $B^T B = I$,

$(AB)^T (A \cdot B) = B^T \cdot \underbrace{A^T \cdot A} \cdot B$

$= B^T \cdot I \cdot B = B^T \cdot B = I$.

- How to show that $O(n,\mathbb{R})$ is a submanifold?

☆ $\Phi : GL(n,\mathbb{R}) \to M(n,\mathbb{R})$
$A \mapsto A^T \cdot A$

so $O(n,\mathbb{R}) = \Phi^{-1}(I)$.

we are going to show that Φ is constant rank.

we use "equivariant const rank thm"

⌈ if $F: M \to N$ maps of G-manifold.
$\circlearrowleft_G \quad \circlearrowleft_G$

and if G acts transitively on M.

then F is a constant rank map. ⌋

if $g \in G = GL(n,\mathbb{R})$, and
$A \rightsquigarrow gA$, then $A^T A \Rightarrow \underbrace{A^T g^T g A}$

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we are going to use right action of G on $GL(n,\mathbb{R})$ and $M(n,\mathbb{R})$

Given ~~an~~ matrix $M \in M(n,\mathbb{R})$

$G \times M(n,\mathbb{R}) \to M(n,\mathbb{R})$

$(g, M) \mapsto g^T M g$

- check: this indeed is a right action.

$(M \cdot g_1) \cdot g_2 = M \cdot (g_1 g_2)$

LHS $= (g_1^T \cdot M \cdot g_1) \cdot g_2$

$= g_2^T \cdot (g_1^T \cdot M \cdot g_1) g_2$

RHS $= (g_1 g_2)^T \cdot M \cdot (g_1 g_2)$

$= g_2^T \cdot g_1^T \cdot M \cdot g_1 \cdot g_2$

This makes $\Phi: GL(n,\mathbb{R}) \to M(n,\mathbb{R})$ equivariant, hence Φ is constant rank.

This week's HW includes
Example 7.29, 7.30

Lemma 7.12 Prop 7.14.

Lemma 7.12

Suppose G: Lie gp
$H \subset G$ open subgp.

$\Rightarrow$ ① H is an embedded Lie subgp.

② H is closed, hence a union of connected components of G.

- let $SO(2)_{i,j} \subset SO(n)$ be the subgp that rotates the ij entry.
- can you prove that the subset $S = SO(2)_{1,2} \cup SO(2)_{2,3} \cup \cdots \cup SO(2)_{n-1,n}$ generate $SO(n)$?

Pf: ① any open submfd is also an embedded submfd

② G is a disjoint union of left coset of H

$gH = \{g \cdot h \mid h \in H\}$. $G =$

$G \setminus H$ is a union of cosets of H, hence is a union of open subsets, hence $G\setminus H$ is open

$\Rightarrow$ H is closed.

• **Def**: Suppose $S \subset G$ any subset, the subgrp $\langle S\rangle$ gen by S is the smallest group in G containing S.

$\langle S \rangle$ contains finite expression involving elements of S, and m, i.

Ex: (1) $G = \mathbb{R}$ (translation)

$S = \{\frac{1}{2}, \frac{1}{3}\}$.

$\langle S \rangle = \frac{1}{6}\cdot \mathbb{Z} \subset G$.

(2) $G = \Sigma_5$ permutation of $\{1,2,3,4,5\}$.

$S = (12), (23), (34), (45)$.

then claim $\langle S \rangle = G$.

7.14

Prop: Suppose G is a Lie group, $W \subset G$ open $e \in W$. Then

(a). W gen an open subgrp of G.

(b). if W is connected, it gen a conn. subgp.

(c) If G is conn., then W gen G.

Pf sketch:

• notations : $A, B \subset G$.

$A\cdot B = \{a\cdot b \mid a \in A, b \in B\} \subset G$

$A^{-1} = \{a^{-1} \mid a \in A\} \subset G$.

• W_k = elements in G can be expressed using $\underline{w_1(w_2 \cdots w_k)}$ where $w_i \in W \cup W^{-1}$.

since w_i can be identity e,

$W_{k-1} \subset W_k$.

• $W_k = W_1 \cdot W_{k-1} = \bigcup_{g \in W_1} L_{g_1}(W_{k-1})$.

① W^{-1} is open. $\because$ $i: G \to G$ is diffeo. W is open.

$W^{-1} \cup W = W_1$ is open

② W_k is open by induction. $\because L_g$ is diffeo. $\therefore L_g(W_{k-1})$ is open

$\langle W \rangle = \bigcup_{k=1}^{\infty} W_k$. Hence is open.

$\langle W\rangle$ is a subgroup, $\because$ $\alpha \in W_k \subset \langle W \rangle$

$\beta \in W_l \subset \langle W \rangle$.

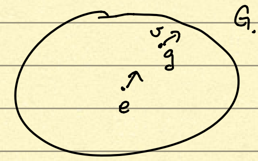
then $\alpha\cdot\beta \in W_{k+l} \subset \langle W \rangle$.

$\alpha^{-1} \in W_k \subset \langle W \rangle$. $\alpha = w_1 \cdots w_k$.

$\alpha^{-1} = w_k^{-1}\cdot w_{k-1}^{-1} \cdots w_1^{-1} \in W_k$.

(b), (c) are skipped.

• Maurer - Cartan 1-form on G.



$v \in T_g G$.

v can be translated back, using $(L_{g^{-1}})_*$ to $T_e G$.

$(L_{g^{-1}})_*(v) \in T_e G = \mathrm{Lie}(G)$

$= \mathfrak{X}(G)^L$

Def: Maurer - Cartan 1-form is a $\mathrm{Lie}(G)$-valued 1-form.

$\omega : TG \to \mathrm{Lie}(G)$.

$(g, v) \mapsto (L_{g^{-1}})_*(v) \in \mathrm{Lie}(G)$.

If G is a matrix group, i.e. $G \subset GL(N, \mathbb{R})$

then $\omega_g = g^{-1}\cdot dg$.

i.e, given a small change to g, $g + \varepsilon h$ ($\in GL(N,\mathbb{R})$).

$g, h \in M(N, \mathbb{R})$.

$\omega_g(h) = g^{-1}\cdot h$.

$g + \varepsilon \underline{h} = g\cdot(I_n + \varepsilon\cdot \underline{\underline{g^{-1}h}})$

• Property of ω : if X are Y are left invariant vector field on G,

we know $X = v^L$, $Y = u^L$ ← $v \in T_e G$. v^L : the left invariant v.f. $v^L|_e = v$.

then $\omega_g(X_g) = v$

$\omega_g(Y_g) = u$.

$\omega([X, Y]) = [\omega(X), \omega(Y)]$.

for arbitrary X, Y. v.f on G.

• $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$.

Now, for X, Y left invariant, we have,

$X(\underline{\omega(Y)}) = 0$, $Y(\underline{\omega(X)}) = 0$

$$d\omega(X,Y) + [\omega(X), \omega(Y)] = 0 \qquad (*)$$

for any X, Y left invariant v.f.

<u>Claim</u>: $(*)$ holds for any vector fields X, Y on G.