Today:

① application: Levi-Civita conn on Lie G.

② geodesic. normal coordinates.

· G: Lie group of dim n.

$\mathfrak{g} := \mathrm{Lie}(G)$ = left-invariant vector fields on G
$= T_eG$.

· a vector space V, and a ~~map~~ Lie group homo. $\rho: G \to GL(V)$ is a representation of G, denote (V,ρ).

· Adjoint representation: $(V,\rho) = (\mathfrak{g}, \mathrm{Ad})$ (of G).

· conj: $G \circlearrowright G$.

$$G \times G \to G$$
$$(g, h) \mapsto g \cdot h \cdot g^{-1}.$$

conj action preserves $e \in G$.

hence acts on T_eG.

$\mathrm{Ad}: G \circlearrowright T_eG = \mathfrak{g}$. ($GL(V)$)

$g \in G$, $X \in \mathfrak{g}$

in matrix form: $\mathrm{Ad}_g(X) = g \cdot X \cdot g^{-1}$ (matrix multiplications)

· Action by on a Lie algebra:

given any Lie group rep (V,ρ), there is an induced Lie algebra rep.

$$\rho: G \to GL(V).$$
$$D\rho_e: \underset{\mathfrak{g}}{\overset{\shortparallel}{T_eG}} \to \underset{\mathrm{End}(V)}{\overset{\shortparallel}{T_{Id}GL(V)}}.$$

$\mathrm{Ad}: G \longrightarrow GL(\mathfrak{g})$. Lie group homomorphism

· $\mathrm{ad}: \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ Lie algebra homomorphism.

in matrix form, if $\mathfrak{g} \subset M_n(\mathbb{R})$.
$X \in \mathfrak{g}$, $Y \in \mathfrak{g}$.
$\mathrm{ad}_X(Y) := [X, Y]$. (matrix commutator.)

$X, Y \in \mathfrak{g}$.

$$[\mathrm{ad}_X, \mathrm{ad}_Y]_{\mathrm{End}(\mathfrak{g})} \overset{\text{Lie alg hom.}}{=} \mathrm{ad}([X,Y]_{\mathfrak{g}})$$

// vector space

· Assume $\mathfrak{g}$ has an inner product that is invariant under Adjoint action by G.

i.e. given $X, Y \in \mathfrak{g}$

$\langle X, Y\rangle = \langle \mathrm{Ad}_g X, \mathrm{Ad}_g Y\rangle$ $\forall g \in G$.

$\Leftrightarrow$ $\forall Z \in \mathrm{Lie}(G) = \mathfrak{g}$.

$$0 = \langle \mathrm{ad}_Z X, Y\rangle + \langle X, \mathrm{ad}_Z Y\rangle.$$

such inner product on $\mathfrak{g}$ is called "conjugation invariant inner product"

We can induce a Left-invariant metric h on G from this conjugation invariant metric on $\mathfrak{g}$.

$\rightsquigarrow$ a bi-invariant metric on G. (invariant under L_g, R_g).

Levi-Civita connection associated to such bi-inv. metric?

Eq (4.1.3). from metric g ~~w~~ to the connection

$$g(\nabla_X Z, Y) = \tfrac{1}{2}\{ X g(Y,Z) + Z g(X,Y) - Y g(Z,X)$$
$$- g([X,Y],Z) + g([Y,Z],X) - g([Z,X],Y)\}$$

Now, let g be the bi-invariant metric h on G. and X, Y, Z be left-invariant vector fields

then. $h(X,Y)$, $h(Y,Z)$, ... are constant.

$h(-,-) = \langle -,- \rangle$.

$$\langle \nabla_X Z, Y\rangle = \tfrac{1}{2}\{ 0 + 0 - 0 - \langle [X,Y], Z\rangle$$
$$+ \langle [Y,Z], X\rangle - \langle [Z,X], Y\rangle\}.$$

$$- \langle [X,Y], Z\rangle + \langle [Y,Z], X\rangle.$$
$$= \langle [Y,X], Z\rangle + \langle [Y,Z], X\rangle.$$
$$= \langle \mathrm{ad}_Y(X), Z\rangle + \langle X, \mathrm{ad}_Y(Z)\rangle \overset{\langle,\rangle \text{ is conjugation invariant}}{=} 0$$

$$\langle \nabla_X Z, Y\rangle = -\tfrac{1}{2}\langle [Z,X], Y\rangle.$$
$$= \langle \tfrac{1}{2}[X,Z], Y\rangle.$$

$\because$ Y is any left-invariant v.f.

$$\boxed{\nabla_X Z = \tfrac{1}{2}[X,Z] \quad (\forall X, Z \text{ left-invariant v.f.})}$$

· We know $TG \cong G \times \mathfrak{g}$
$\cong G \times T_eG$.
is a trivial bundle.

· For any trivial vector bundle, and fix a trivialization, $\{e_\alpha\} \in \Gamma(M,E)$ we can associate it in a trivial connection d, s.t. $d.e_\alpha = 0$.

· Ex 4.1.20. (to see Levi-Civita connection is between the left-flat-connection and right-flat-connection).

Geodesic equation:

(M, g) Riem mfd

a curve $\gamma: (a, b) \to M$ is a geodesic, if

$$\nabla_{\dot\gamma(t)} (\dot\gamma(t)) = 0. \qquad (*)$$

local existence & uniqueness result:

$\forall\, p \in M, \quad \forall\, X_p \in T_pM.$

$\exists$

$\gamma: (-\varepsilon, \varepsilon) \to M.$

s.t. $\gamma(0) = p$, $\dot\gamma(0) = X_p.$

$x^1, \cdots, x^n$

In local chart, the geodesic eqn:

$$(*) \quad \ddot{x}^i(t) + \Gamma^i_{jk}(x(t)) \cdot \dot{x}^j(t) \cdot \dot{x}^k(t) = 0.$$

$x(t) = \gamma(t) = (x^1(t), \cdots, x^n(t)).$

$$A_i \cdot \dot\gamma^i(t) = A(\dot\gamma(t))$$

geodesic complete space:

any geodesic can exist for arbitrary long time.