

3.

$$\tau = 2\pi.$$

$\xi > 0$:

We integrate over a negatively oriented semicircle in the closed lower half-plane:

$$\int_{\gamma_R} \frac{a}{a^2 + x^2} e^{-\tau i x \xi} dx = \int_{-R}^R \frac{a}{a^2 + x^2} e^{-\tau i x \xi} dx + \int_{\theta=0}^{-\tau/2} \frac{a}{a^2 + z^2} e^{-\tau i z \xi} dz$$

where $z = R e^{i\theta}$ in the second integral.

$|e^{-\tau i z \xi}| \leq 1$, so the second integral has magnitude bounded above by

$$\frac{a^{\frac{\tau}{2}} R}{R^2 - a^2}$$

hence the integral over γ_R approaches

$$\int_{-\infty}^{\infty} \frac{a}{a^2 + x^2} e^{-\tau i x \xi} dx$$

as $R \rightarrow \infty$.

Furthermore, for sufficiently large R we have by Cauchy's integral formula

$$\begin{aligned} \frac{1}{\pi} \int_{\gamma_R} (\dots) &= \frac{1}{\pi} - 2\pi i \frac{a}{-2ia} e^{-\tau i(-ia)\xi} \\ &= e^{-\tau a|\xi|} \end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} (\dots) = e^{-\tau a|\xi|}$$

$\xi < 0$:

Similar, but with a positively oriented semicircle in the closed upper half-plane, giving the same result:

$$\int_{-\infty}^{\infty} (\dots) = e^{-\tau a|\xi|}$$

$\xi = 0$:

$$\begin{aligned} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{a^2 + x^2} e^{-\tau i x \xi} dx &= \frac{1}{\pi} \lim_{R \rightarrow \infty} [\tan^{-1}(a^{-1}x)]_{-R}^R \\ &= \frac{1}{\pi} \pi \\ &= 1 \\ &= e^{-\tau a|\xi|} \end{aligned}$$

Inversion:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\tau a|\xi|} e^{\tau i x \xi} d\xi &= \int_{-\infty}^0 e^{\tau a \xi} e^{-\tau i x \xi} d\xi + \int_0^{\infty} e^{-\tau a \xi} e^{\tau i x \xi} d\xi \\ &= \frac{1}{\tau(a + ix)} + \frac{1}{\tau(a - ix)} \\ &= \frac{1}{\pi} \frac{a}{a^2 + x^2} \end{aligned}$$

4.

$\xi \geq 0$:

Writing

$$f(z) = \frac{e^{-\tau iz\xi}}{Q(z)}$$

we integrate over a negatively oriented semicircle γ_R in the closed lower half-plane:

$$\int_{\gamma_R} f(z)dz = \int_{-R}^R f(x)dx + \int_{|z|=R, \text{Im } z \leq 0} f(z)dz$$

The latter integral has magnitude bounded above by

$$\frac{A\pi R}{R^{\deg Q}}$$

for some $A > 0$, and hence it $\rightarrow 0$ as $R \rightarrow \infty$.

Therefore

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R \rightarrow \infty} \int_{\gamma_R} f(z)dz$$

For sufficiently large R ,

the singularities of f inside γ_R

are poles and are precisely the zeros of Q in the lower half-plane.

Residue formula gives

$$\int_{-\infty}^{\infty} f(x)dx = -2\pi i \sum_{r \in R \cap \{\text{Im } z < 0\}} \text{res}_r f$$

where $R = Q^{-1}(\{0\})$.

For $r \in R$, let m_r be the multiplicity of r . We have

$$Q(z) = c \prod_{r \in R} (z - r)^{m_r}$$

for some $c \neq 0$.

Defining

$$f_r(z) = \frac{e^{-\tau iz\xi}}{c \prod_{s \in R \setminus \{r\}} (z - s)^{m_s}}$$

we find that $\text{res}_r f$

is the $(m_r - 1)$ -th coefficient of the power series for f_r centered at r , since

$$f(z) = \frac{f_r(z)}{(z - r)^{m_r}}$$

Therefore

$$\begin{aligned} \text{res}_r f &= \frac{1}{(m_r - 1)!} f_r^{(m_r - 1)}(r) \\ \int_{-\infty}^{\infty} f(x)dx &= -2\pi i \sum_{r \in R \cap \{\text{Im } z < 0\}} \frac{1}{(m_r - 1)!} f_r^{(m_r - 1)}(r) \end{aligned}$$

$\xi \leq 0$:

Similar, but with a positively oriented semicircle in the closed upper half-plane, giving

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{r \in R \cap \{\text{Im } z > 0\}} \frac{1}{(m_r - 1)!} f_r^{(m_r-1)}(r)$$

When several roots coincide:

When several roots coincide at r , the expression

$$\text{res}_r f = \frac{1}{(m_r - 1)!} f_r^{(m_r-1)}(r)$$

uses a higher-order derivative of f_r .

Whereas when r is simple, we have

$$\text{res}_r f = f_r(r)$$

6.

Poisson's summation formula and exercise 3 give

$$\begin{aligned}\frac{1}{\pi} \sum_{-\infty < n}^{< \infty} \frac{a}{a^2 + n^2} &= \sum_{-\infty < n}^{< \infty} e^{-2\pi a|n|} \\ &= 2 \sum_{0 \leq n}^{< \infty} e^{-2\pi a n} - 1 \\ &= \frac{2}{1 - e^{-2\pi a}} - 1 \\ &= \frac{2}{1 - e^{-2\pi a}} - \frac{1 - e^{-2\pi a}}{1 - e^{-2\pi a}} \\ &= \frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}} \\ &= \frac{\frac{1}{2}(e^{\pi a} + e^{-\pi a})}{\frac{1}{2}(e^{\pi a} - e^{-\pi a})} \\ &= \frac{\cosh \pi a}{\sinh \pi a} \\ &= \coth \pi a\end{aligned}$$

8.

$\tau = 2\pi$.

Integration by parts gives

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)e^{-\tau ix\xi} dx &= -\int_{-\infty}^{\infty} f'(x) \left(-\frac{1}{\tau i\xi}\right) e^{-\tau ix\xi} dx \\ &= \frac{1}{\tau i\xi} \widehat{f}'(\xi)\end{aligned}$$

and shows the convergence of the latter integral.

Applying this repeatedly yields

$$\widehat{f^{(n)}}(\xi) = (\tau i\xi)^n \widehat{f}(\xi)$$

Using Fourier inversion,

$$\begin{aligned}f^{(n)}(x) &= \int_{-M}^M \widehat{f^{(n)}}(\xi) e^{\tau ix\xi} d\xi \\ &= (\tau i)^n \int_{-M}^M \widehat{f}(\xi) \xi^n e^{\tau ix\xi} d\xi\end{aligned}$$

$$\begin{aligned}a_n &= \frac{1}{n!} f^{(n)}(0) \\ &= \frac{(\tau i)^n}{n!} \int_{-M}^M \widehat{f}(\xi) \xi^n d\xi\end{aligned}$$

Next we have

$$\begin{aligned}(n!|a_n|)^{1/n} &\leq \tau \left| \int_{-M}^M \widehat{f}(\xi) \xi^n d\xi \right|^{1/n} \\ &\leq \tau \left| 2M \left(\max_{[-M,M]} |\widehat{f}| \right) M^n \right|^{1/n} \\ &\leq \tau M \left| 2M \left(\max_{[-M,M]} |\widehat{f}| \right) \right|^{1/n} \\ &\leq \tau M \left| 2M \left(\max \left\{ 1, \max_{[-M,M]} |\widehat{f}| \right\} \right) \right|^{1/n} \\ &\rightarrow \tau M\end{aligned}$$

as $n \rightarrow \infty$. So

$$\limsup_{n \rightarrow \infty} (n!|a_n|)^{1/n} \leq \tau M$$

For the converse direction, we first determine easily that f is holomorphic in the complex plane, since the lim sup bound shows that the radius of convergence of the power series is infinity.

Let $\varepsilon > 0$.

Let $N \in \mathbb{N}$ be large enough that

$$\begin{aligned} n \geq N &\implies (n!|a_n|)^{1/n} \leq \tau M + \tau\varepsilon = \tau(M + \varepsilon) \\ &\implies |a_n| \leq \frac{1}{n!}(\tau[M + \varepsilon])^n \\ &\implies |a_n||z|^n \leq \frac{1}{n!}[\tau(M + \varepsilon)|z|]^n \quad \forall z \in \mathbb{C} \end{aligned}$$

Then for any $z \in \mathbb{C}$,

$$\begin{aligned} \left| \sum_{N \leq n}^{\infty} a_n z^n \right| &\leq \sum_{N \leq n}^{\infty} |a_n||z|^n \\ &\leq \sum_{N \leq n}^{\infty} \frac{1}{n!}[\tau(M + \varepsilon)|z|]^n \end{aligned}$$

Furthermore, as $z \rightarrow \infty$ we have

$$\frac{\sum_{0 \leq n}^{\infty} a_n z^n}{\sum_{0 \leq n}^{\infty} \frac{1}{n!}[\tau(M + \varepsilon)|z|]^n} \rightarrow \frac{a_{N-1}}{\frac{1}{(N-1)!}[\tau(M + \varepsilon)]^{N-1}}$$

Hence for some $A, R > 0$ we have

$$|z| > R \implies \left| \sum_{0 \leq n}^{\infty} a_n z^n \right| \leq A \sum_{0 \leq n}^{\infty} \frac{1}{n!}[\tau(M + \varepsilon)|z|]^n$$

Since $\sum_{0 \leq n}^{\infty} \frac{1}{n!}[\tau(M + \varepsilon)|z|]^n$ is nonzero everywhere, we also have

$$|z| \leq R \implies \left| \sum_{0 \leq n}^{\infty} a_n z^n \right| \leq B \sum_{0 \leq n}^{\infty} \frac{1}{n!}[\tau(M + \varepsilon)|z|]^n$$

For some $B > 0$.

Letting

$$A_\varepsilon = \max\{A, B, 1\}$$

we have at every $z \in \mathbb{C}$,

$$\begin{aligned} \left| \sum_{0 \leq n}^{\infty} a_n z^n \right| &\leq A_\varepsilon \sum_{0 \leq n}^{\infty} \frac{1}{n!}[\tau(M + \varepsilon)|z|]^n \\ \left| \sum_{N \leq n}^{\infty} a_n z^n \right| &\leq A_\varepsilon \sum_{N \leq n}^{\infty} \frac{1}{n!}[\tau(M + \varepsilon)|z|]^n \end{aligned}$$

$$\begin{aligned} |f(z)| &\leq \left| \sum_{0 \leq n < N} a_n z^n \right| + \left| \sum_{N \leq n < \infty} a_n z^n \right| \\ &\leq A_\varepsilon \sum_{0 \leq n < \infty} \frac{1}{n!} [\tau(M + \varepsilon)|z|]^n \\ &= A_\varepsilon e^{\tau(M + \varepsilon)|z|} \end{aligned}$$

QED.

9. (a)

$$\tau = 2\pi.$$

We use the principal branch of the logarithm to exponentiate.

Also $\arg z = \text{Im} \log z$.

First assume $0 < \gamma < 1$.

Define

$$S = \{z \in \mathbb{C} \mid |\arg z| < \gamma \frac{\tau}{4}\}$$

$$G(z) = F(z^{1/\gamma})$$

Then G is holomorphic on S and continuous on \bar{S} .

(S is a wedge with angle $\gamma \frac{\tau}{2}$.)

Pick an α with

$$1 < \alpha < \beta < 1/\gamma$$

for some β .

Then for $z \in S$,

$$|\arg z^\alpha| = \alpha |\arg z| < \beta \gamma \frac{\tau}{4} < \frac{\tau}{4}$$

Hence $\cos(\arg z^\alpha) > \cos(\beta \gamma \frac{\tau}{4}) > 0$.

Defining

$$G_\varepsilon(z) = G(z) e^{-\varepsilon z^\alpha}$$

we have $|G_\varepsilon(z)| \leq C e^{c|z| - \varepsilon \cos(\beta \gamma \frac{1}{4}) |z|^\alpha}$.

This bound approaches 0 as $|z| \rightarrow \infty$.

For any R large enough that $C e^{cR - \varepsilon \cos(\beta \gamma \frac{1}{4}) R^\alpha} \leq 1$,

we find that $|G(z)| \leq 1$ throughout the boundary of the wedge $\overline{D_R(0)} \cap S$, hence also throughout the wedge.

Since R was an arbitrary large radius,

we see that $|G| \leq 1$ on S .

Now for each z in the right half-plane, $F(z) = G(z^\gamma)$ hence $|F(z)| \leq 1$.

This proves the case $0 < \gamma < 1$.

If $\gamma \leq 0$, then pick $C' \geq C$ with

$$|z| < 1 \implies |F(z)| \leq C'$$

Letting $\gamma' \in (0, 1)$, we see that

$$|z| < 1 \implies |F(z)| \leq C' e^{c|z|^{\gamma'}}$$

$$|z| \geq 1 \implies |F(z)| \leq C e^{c|z|^\gamma}$$

$$\implies |F(z)| \leq C' e^{c|z|^{\gamma'}}$$

Hence $|F(z)| \leq C' e^{c|z|^{\gamma'}}$ throughout the right half-plane, so $|F| \leq 1$ by the previous argument.