1.

Complex zeros:

For any zero z of $\sin \pi z$, we have

$$0 = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$$
$$e^{i\pi z} = e^{-i\pi z}$$
$$e^{i2\pi z} = 1$$

Letting $z = x + iy$,

$$e^{-2\pi y} e^{-i2\pi x} = 1$$
$$\left| e^{-2\pi y} \right| = 1$$
$$y = 0$$

So all zeros are real.
Our knowledge of $x \mapsto \sin \pi x$ as a real-to-real function tells us that its real zeros are precisely the integers.
Hence the zeros of the complex-to-complex function $z \mapsto \sin \pi z$ are also precisely the integers.

Order:

Write

$$\sin \pi z = \sum_{0 \le k}^{<\infty} a_k (z-n)^k$$

and note that

$$a_0 = \sin \pi n = 0$$
$$a_1 = (\cos \pi n)\pi = \pm\pi \neq 0$$

Therefore the function

$$g(z) = \frac{1}{\sum_{1 \le k}^{<\infty} a_k (z-n)^{k-1}}$$

is holomorphic on a neighborhood of n,
nonvanishing at n,
and satisfies

$$\frac{1}{\sin \pi z} = (z-n)^{-1} g(z)$$

on a deleted neighborhood of n,
hence the pole at n has order 1.

Residue: $(-1)^n \pi$.

Proof:

Letting g be as above
and writing

$$g(z) = \sum_{0 \le k}^{<\infty} b_k (z-n)^k$$

we have

$$\operatorname{res}_n f = b_0$$

since the pole is of order 1.
Hence

$$\begin{aligned}
\operatorname{res}_n f &= b_0 \\
&= g(n) \\
&= \frac{1}{\sum_{1 \le k}^{<\infty} a_k (n-n)^{k-1}} \\
&= \frac{1}{a_1 (n-n)^{1-1}} \\
&= \frac{1}{a_1}
\end{aligned}$$

Recall that a_k are the coefficients
of the series for $\sin \pi z$
centered at n.
Hence

$$a_1 = (\cos \pi n)\pi = (-1)^n \pi$$

$$\operatorname{res}_n f = (-1)^n \pi$$

QED.

2.

Poles: $\frac{\pm 1 \pm i}{\sqrt{2}}$

Proof:

Since $1 + z^4$ has degree 4, it has at most 4 zeros.
Computation shows that the 4 numbers described above are zeros,
hence the zeros of $1 + z^4$ are *precisely* those numbers.
hence the roots of $\frac{1}{1+z^4}$ are precisely those numbers.

Integral: $\frac{\pi}{\sqrt{2}}$.

Proof:

For convenience, define two families of contours $[0, 1] \to \mathbb{C}$:

$$\alpha_R(t) = -R + 2Rt$$

$$\beta_R(t) = Re^{i\pi(t)}$$

(Diameter and semicircle, respectively.)
This lets us define a family
of positively oriented semicircle-shaped contours $[0, 1] \to \mathbb{C}$:

$$\gamma_R(t) = \begin{cases} \alpha_R(2t) & t \in [0, \frac{1}{2}] \\ \beta_R(2t - 1) & t \in [\frac{1}{2}, 1] \end{cases}$$

Evaluation by residue formula:

For sufficiently large R,
the contour γ_R encloses precisely the following poles of $\frac{1}{1+z^4}$:

$$\frac{-1 + i}{\sqrt{2}} \text{ and } \frac{+1 + i}{\sqrt{2}}$$

Each pole has order 1 since

$$\frac{1}{1 + z^4} = \frac{1}{z - \frac{+1+i}{\sqrt{2}}} \frac{1}{z - \frac{-1+i}{\sqrt{2}}} \frac{1}{z - \frac{-1-i}{\sqrt{2}}} \frac{1}{z - \frac{+1-i}{\sqrt{2}}}$$

This gives

$$\begin{aligned} \int_{\gamma_R} \frac{1}{1 + z^4} dz =& 2\pi i \operatorname{res}_{\frac{-1+i}{\sqrt{2}}} \left(\frac{1}{1 + z^4} \right) \\ &+ 2\pi i \operatorname{res}_{\frac{+1+i}{\sqrt{2}}} \left(\frac{1}{1 + z^4} \right) \end{aligned}$$

where the residue at $z_1 = \frac{-1+i}{\sqrt{2}}$ is found by evaluating

$$\frac{1}{z_1 - \frac{+1+i}{\sqrt{2}}} \frac{1}{z_1 - \frac{-1-i}{\sqrt{2}}} \frac{1}{z_1 - \frac{+1-i}{\sqrt{2}}}$$

and the reside at $z_2 = \frac{+1+i}{\sqrt{2}}$ is found by evaluating

$$\frac{1}{z_2 - \frac{-1+i}{\sqrt{2}}} \frac{1}{z_2 - \frac{-1-i}{\sqrt{2}}} \frac{1}{z_2 - \frac{+1-i}{\sqrt{2}}}$$

This gives

$$\begin{aligned}
\int_{\gamma_R} \frac{1}{1+z^4} dz =& 2\pi i \left(\frac{1}{8(1+i)/(\sqrt{2})^3} + \frac{1}{8(-1+i)/(\sqrt{2})^3} \right) \\
=& 2\pi i \frac{1}{8} (\sqrt{2})^3 \left(\frac{1}{1+i} + \frac{1}{-1+i} \right) \\
=& 2\pi i \frac{1}{8} (\sqrt{2})^3 (-i) \\
=& 2\pi \frac{1}{8} (\sqrt{2})^3 \\
=& \frac{\pi}{\sqrt{2}}
\end{aligned}$$

Equating to desired integral:

First note that when $|z| = R$,

$$|1 + z^4| \geq R^4 - 1$$

$$|\frac{1}{1+z^4}| \leq \frac{1}{R^4 - 1}$$

Second, note that the semicircle β_R has arclength πR.
Therefore

$$\left| \int_{\beta_R} \frac{1}{1+z^4} dz \right| \leq \frac{\pi R}{R^4 - 1}$$

which approaches 0 as $R \to \infty$.
Hence

$$\begin{aligned}
\frac{\pi}{\sqrt{2}} &= \lim_{R\to\infty} \frac{\pi}{\sqrt{2}} \\
&= \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{1+z^4} \\
&= \lim_{R\to\infty} \int_{\alpha_R} \frac{1}{1+z^4} dz + \lim_{R\to\infty} \int_{\beta_R} \frac{1}{1+z^4} dz \\
&= \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx \quad + 0 \\
&= \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx
\end{aligned}$$

QED.

3.

One trick and one contour.

Trick:

$\frac{\sin x}{x^2+a^2}$ is an odd function,
so its integral from $-\infty$ to $+\infty$ is 0,
so

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}dx &= \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x^2+a^2}dx \\ &= \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}dx\end{aligned}$$

Contour:

Same as in exercise 2.
For convenience, define two families of contours $[0,1] \to \mathbb{C}$:

$$\alpha_R(t) = -R + 2Rt$$

$$\beta_R(t) = Re^{i\pi(t)}$$

(Diameter and semicircle, respectively.)
This lets us define a family
of positively oriented semicircle-shaped contours $[0,1] \to \mathbb{C}$:

$$\gamma_R(t) = \begin{cases} \alpha_R(2t) & t \in [0, \frac{1}{2}] \\ \beta_R(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

Evaluating contour integral:

We have

$$\frac{e^{iz}}{z^2+a^2} = \frac{e^{iz}/(z+ia)}{z-ia}$$

The numerator is holomorphic everywhere except $-ia$;
in particular, it is holomorphic on γ_R and the region it encloses.

Furthermore, for sufficiently large R,
γ_R encloses the point ia,
giving (by Cauchy's integral formula):

$$\begin{aligned}\int_{\gamma_R} \frac{e^{iz}}{z^2+a^2}dz &= \int_{\gamma_R} \frac{e^{iz}/(z+ia)}{z-ia}dz \\ &= 2\pi i \cdot e^{iia}/(ia+ia) \\ &= \pi\frac{e^{-a}}{a}\end{aligned}$$

Equating to desired integral:

First we show that

$$\int_{\beta_R} \frac{e^{iz}}{z^2+a^2}dz \quad \to \quad 0$$

as $R \to \infty$.
Observe that, with $z = x + iy$, we have $y \geq 0$ on β_R.

Therefore

$$\begin{aligned}
\left|\frac{e^{iz}}{z^2+a^2}\right| &= \left|e^{-y}e^{ix}\frac{1}{z^2+a^2}\right| \\
&= \left|e^{-y}\frac{1}{z^2+a^2}\right| \\
&\le \left|\frac{1}{z^2+a^2}\right| \\
&\le \frac{1}{R^2-a^2}
\end{aligned}$$

Since the semicircle β_R has arclength πR, this gives

$$\left|\int_{\beta_R}\frac{e^{iz}}{z^2+a^2}\right| \le \frac{\pi R}{R^2-a^2}$$

which approaches 0, as desired.

This gives

$$\begin{aligned}
\pi\frac{e^{-a}}{a} &= \lim_{R\to\infty}\frac{e^{-a}}{a} \\
&= \lim_{R\to\infty}\int_{\alpha_R}\frac{e^{iz}}{z^2+a^2}dz \quad + \lim_{R\to\infty}\int_{\beta_R}\frac{e^{iz}}{z^2+a^2}dz \\
&= \int_{-\infty}^{\infty}\frac{e^{ix}}{z^2+a^2}dx \\
&= \int_{-\infty}^{\infty}\frac{\cos x}{z^2+a^2}dx
\end{aligned}$$

QED.

7.

We interpret θ as the angle of a unit-length complex number z.
The integrand becomes

$$\begin{aligned}\frac{d\theta}{(a+\cos\theta)^2} &= \frac{dz/(iz)}{(a+\frac{1}{2}(z+z^{-1}))^2}\\ &= \frac{-i4z}{(z^2+2a+1)^2}dz\end{aligned}$$

The denominator factors as

$$\begin{aligned}z^2+2a+1 &= (z+a)^2-(a^2-1)\\ &= (z+a-\sqrt{a^2-1})(z+a+\sqrt{a^2-1})\\ &= (z-(\sqrt{a^2-1}-a))(z-(-\sqrt{a^2-1}-a))\end{aligned}$$

Note magnitudes:
$\left|\sqrt{a^2-1}-a\right| < 1$ because $-1 < \sqrt{a^2-1}-a < 0$ because $a > 1$.
$\left|-\sqrt{a^2-1}-a\right| > 1$ because $-\sqrt{a^2-1}-a < -1$ because $a > 1$.
So,
to use Cauchy's integral formula, we write our integrand as

$$\frac{-i4z/\left(z-(-\sqrt{a^2-1}-a)\right)^2}{\left(z-(\sqrt{a^2-1}-a)\right)^2}dz$$

Defining $f(z)$ to be the numerator, we have
(writing + above the integral sign to indicate positive orientation):

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2} &= \int_{|z|=1}^{+}\frac{f(z)}{\left(z-(\sqrt{a^2-1}-a)\right)^2}dz\\ &= 2\pi i\cdot f'(\sqrt{a^2-1}-a)\end{aligned}$$

Noting that

$$f'(z) = -i4\frac{-z+a+\sqrt{a^2-1}}{\left(z+a+\sqrt{a^2-1}\right)^3}$$

we compute

$$f'(\sqrt{a^2-1}-a) = \frac{a}{i\,(a^2-1)^{3/2}}$$

and continue

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2} &= 2\pi i\cdot f'(\sqrt{a^2-1}-a)\\ &= 2\pi i\cdot\frac{a}{i\,(a^2-1)^{3/2}}\\ &= \frac{2\pi a}{(a^2-1)^{3/2}}\end{aligned}$$

QED.

hw5.5

Let P, Q be polynomials. with $\deg Q \leq \deg P - 2$.
WLOG let both be monic, and write

$$Q(z) = \prod_{j=1}^{m} (z - u_j)$$

$$P(z) = \prod_{j=1}^{n} (z - z_j)$$

with $|z_j| < 1$, as described in the problem.

Define

$$g(w) = \begin{cases} Q(w^{-1})/P(w^{-1}) & w \neq 0 \\ 0 & w = 0 \end{cases}$$

Clearly g is holomorphic on $\mathbb{C} \setminus 0$.
We will show that it is holomorphic at 0 and that $g'(0) = 0$,
enabling the following argument:

$$\begin{aligned} \int_{|z|=1}^{+} \frac{Q(z)}{P(z)} dz &= \int_{|w|=1}^{-} \frac{-w^{-2} Q(w^{-1})}{P(w^{-1})} dw \\ &= \int_{|w|=1}^{+} \frac{Q(w^{-1})/P(w^{-1})}{w^2} dw \\ &= \int_{|w|=1}^{+} \frac{g(w)}{(w-0)^2} dw \\ &= 2\pi i \cdot g'(0) \\ &= 2\pi i \cdot 0 \\ &= 0 \end{aligned}$$

(The + or − above an integral indicates orientation of the circle.)
Here we have substituted $w = z^{-1}$
and applied Cauchy's formula.

To show $g'(0)$ exists and is 0, we note

$$\begin{aligned} \frac{g(h) - g(0)}{h - 0} &= \frac{g(h)}{h} \\ &= \frac{Q(h^{-1}}{hP(h^{-1}} \\ &= \frac{h^{n-1} Q(h^{-1}}{h^n P(h^{-1}} \\ &= \frac{h^{n-1}(h^{-1} - u_1) \cdots (h^{-1} - u_m)}{h^n (h^{-1} - z_1) \cdots (h^{-1} - z_n)} \\ &= \frac{h^{n-m-1}(1 - hu_1) \cdots (1 - hu_m)}{(1 - hz_1) \cdots (1 - hz_n)} \end{aligned}$$

As $h \to 0$, we have

$$\begin{aligned}\lim_{h\to 0}\frac{g(h)-g(0)}{h-0} &= \lim_{h\to 0}\frac{h^{n-m-1}(1-hu_1)\cdots(1-hu_m)}{(1-hz_1)\cdots(1-hz_n)}\\ &= \frac{0^{n-m-1}1\cdots 1}{1\cdots 1}\\ &= \frac{0\cdot 1\cdots 1}{1\cdots 1}\\ &= 0\end{aligned}$$

(The equality $0^{n-m-1} = 0$ uses the fact that $m \le n-2$.)
Hence g is holomorphic at 0 with

$$g'(0) = 0$$

completing the proof.