**10.**

$$
\begin{aligned}
4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} &= (\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y})(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}) \\
&= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{i}\frac{\partial}{\partial x\partial y} + \frac{1}{i}\frac{\partial}{\partial y\partial x} \\
&= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta \\
&= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{i}\frac{\partial}{\partial x\partial y} - \frac{1}{i}\frac{\partial}{\partial y\partial x} \\
&= (\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y})(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}) \\
&= 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}
\end{aligned}
$$

**11.**

For $f$ holomorphic, we have

$$\frac{\partial f}{\partial \bar{z}} = 0$$

And the linearity of $\Delta$ gives

$$\begin{aligned}\Delta u + i\Delta v &= \Delta f \\ &= 4\frac{\partial^2 f}{\partial z \partial \bar{z}} \\ &= 4\frac{\partial 0}{\partial z} \\ &= 0\end{aligned}$$

Comparing real and imaginary parts gives

$$\Delta u = 0$$

$$\Delta v = 0$$

**13.**

**Additional hypothesis:** $\Omega$ *is connected.*

**(a)**

Put $f = u + iv$.
Constancy of $u$ shows that $\forall z \in \Omega :$

$$0 = \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$$
$$0 = \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$$

For any $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$,
in $\Omega$ we now have

$$\begin{aligned} v(z_1) &= v(x_1 + iy_1) \\ &= v(x_1 + iy_1) + 0 + 0 \\ &= v(x_1 + iy_1) + \int_{y=y_1}^{y_2} 0 dy + \int_{x=x_1}^{x_2} 0 dx \\ &= v(x_1 + iy_1) + \int_{y=y_1}^{y_2} \frac{\partial v(x_1 + iy)}{\partial y} dy + \int_{x=x_1}^{x_2} \frac{\partial v(x + iy_2)}{\partial x} dx \\ &= v(x_1 + iy_1) + [v(x_1 + iy_2) - v(x_1 + iy_1)] + [v(x_2 + iy_2) - v(x_1 + iy_2)] \\ &= v(x_2 + iy_2) \\ &= v(z_2) \end{aligned}$$

assuming the domains of integration are in $\Omega$.

Applying this reasoning finitely many times,
we find that any $z_1, z_2 \in \Omega$
connected by a finite chain of vertical and horizontal line segments in $\Omega$
have the same output under $v$.

Since $\Omega$ is open and connected,
any $z_1, z_2 \in \Omega$ have such a connection,
and therefore $v$ is constant on $\Omega$.

Since $f = u + iv$ and $u, v$ are constant, $f$ is constant.

**(b)**

The function

$$\frac{1}{i} f$$

is also holomorphic on $\Omega$ and has constant real part.
By part (a), $\frac{1}{i} f$ is constant, hence $f$ is constant.

**(c)**

**Claim 1:**

$$|f| \text{ constant on } \Omega \implies f' = 0 \text{ on } \Omega$$

**Proof:**

Proof by contraposition.
Assume $f' \neq 0$.
Let $z \in \Omega$ with $f'(z) \neq 0$ and write

$$f(z) = |f(z)|u$$

with $|u| = 1$.
Let $r > 0$ and defining

$$\Delta z = r\frac{u}{f'(z)}$$

Let $r$ be small enough that $z + \Delta z \in \Omega$.
We have

$$\begin{aligned} f(z + \Delta z) &= f(z) + f'(z)\Delta z + R(\Delta z) \\ &= u|f(z)| + ur + R(\Delta z) \\ &= u(|f(z)| + r) + R(\Delta z) \end{aligned}$$

$$u(f(z) + r) = f(z + \Delta z) - R(\Delta z)$$

where $R(\Delta z)$ is the Taylor remainder.
When $r$ is small enough that $\left|\frac{R(\Delta z)}{\Delta z}\right| < |f'(z)|$, we have

$$\begin{aligned} |f(z)| + r &\leq |f(z + \Delta z)| + |R(\Delta z)| \\ &< |f(z + \Delta z)| + r \end{aligned}$$

$$|f(z)| < |f(z + \Delta z)|$$

hence $f$ is nonconstant.
We have now proven

$$f' \neq 0 \text{ on } \Omega \implies |f| \text{ nonconstant on } \Omega$$

giving the contrapositive

$$|f| \text{ constant on } \Omega \implies f' = 0 \text{ on } \Omega$$

as desired.

**Claim 2:**

$$f' = 0 \text{ on } \Omega \implies f \text{ constant on } \Omega$$

**Proof:**

$$\begin{aligned} 0 &= f' \\ &= \frac{\partial f}{\partial x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \end{aligned}$$

Hence, comparing real and imaginary parts,

$$\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = 0$$

Hence, by the integration argument in part (a),

$$v \text{ is constant.}$$

Hence, by part (b),

$$f \text{ is constant.}$$

Combining the two claims gives

$$|f| \text{ constant on } \Omega \implies f \text{ constant on } \Omega$$

**15.**

**Intro.**

We will show that the difference

$$\sum_{\leq n}^{<\infty} a_n - \sum_{\leq n}^{<\infty} r^n a_n$$

approaches 0 as real $r$ approaches 1 from below.

**Notation.**

Define, for any $T \in \mathbb{N}$,

$$S_T = \sum_{1\leq n}^{<T} a_n$$

$$E_T = \sum_{T\leq n}^{<\infty} a_n$$

$$M_T = \max_{1\leq n}^{<T} |a_n|$$

**Lemma.** Let $r < 1, N \in \mathbb{N}$.

$$\sum_{N\leq n}^{<\infty} r^n a_n = r^N E_N + \sum_{N+1\leq n}^{<\infty} (r^n - r^{n-1}) E_n$$

**Proof:**

$$\begin{aligned}
\sum_{N\leq n}^{\infty} r^n a_n &= r^N E_N + \sum_{N+1\leq n}^{<\infty} (r^n - r^N) a_n \\
&= r^N E_N + (r^{N+1} - r^N) E_{N+1} + \sum_{N+2\leq n}^{<\infty} (r^n - r^{N+1}) a_n \\
&= r^N E_N + (r^{N+1} - r^N) E_{N+1} + (r^{N+2} - r^{N+1}) E_{N+2} + \sum_{N+3\leq n}^{<\infty} (r^n - r^{N+2}) a_n \\
&= r^N E_N + \sum_{N+1\leq n}^{<N+k} (r^n - r^{n-1}) E_n + \sum_{N+k\leq n}^{<\infty} (r^n - r^{N+k-1}) a_n
\end{aligned}$$

Subtracting,

$$\begin{aligned}
\sum_{N\leq n}^{\infty} a_n - \left( r^N E_N + \sum_{N+1\leq n}^{<N+k} (r^n - r^{n-1}) E_n \right) &= \sum_{N+k\leq n}^{<\infty} (r^n - r^{N+k-1}) a_n \\
&= \sum_{N+k\leq n}^{<\infty} r^n a_n - r^{N+k-1} \sum_{N+k\leq n}^{<\infty} a_n
\end{aligned}$$

Taking the limit as $k \to \infty$,

$$\sum_{N\leq n}^{\infty} a_n - \left( r^N E_N + \sum_{N+1\leq n}^{<\infty} (r^n - r^{n-1}) E_n \right) = 0$$

which gives the desired equality.

*End proof of lemma.*

**Bridge.**

Now we point out, for any $r < 1$ and $N \in \mathbb{N}$:

$$\begin{aligned}
\sum_{1\le n}^{<\infty} a_n - \sum_{1\le n}^{<\infty} r^n a_n &= \sum_{1\le n}^{<\infty} (1-r^n)a_n \\
&= \sum_{1\le n}^{<N} (1-r^n)a_n + \sum_{N\le n}^{<\infty} a_n - \sum_{N\le n}^{<\infty} r^n a_n \\
&= \sum_{1\le n}^{<N} (1-r^n)a_n + E_N - \left( r^N E_N + \sum_{N+1\le n}^{<\infty} (r^n - r^{n-1})E_n \right)
\end{aligned}$$

**Core.**

Let $\varepsilon > 0$.

Choosing $N$ large enough that $n > N \implies |E_n| < \frac{1}{4}\varepsilon$, we have

$$\begin{aligned}
\left| \sum_{1\le n}^{<\infty} a_n - \sum_{1\le n}^{<\infty} r^n a_n \right| &\le \left| \sum_{1\le n}^{<N} (1-r^n)a_n \right| + |E_N| + r^N |E_N| + \sum_{N+1\le n}^{<\infty} \left| r^n - r^{n-1} \right| |E_n| \\
&< \left| \sum_{1\le n}^{<N} (1-r^n)a_n \right| + \frac{1}{4}\varepsilon + r^N \frac{1}{4}\varepsilon + \sum_{N+1\le n}^{<\infty} \left| r^n - r^{n-1} \right| \frac{1}{4}\varepsilon \\
&= \left| \sum_{1\le n}^{<N} (1-r^n)a_n \right| + \frac{1}{4}\varepsilon + r^N \frac{1}{4}\varepsilon + \sum_{N+1\le n}^{<\infty} (r^{n-1} - r^n) \frac{1}{4}\varepsilon \\
&= \left| \sum_{1\le n}^{<N} (1-r^n)a_n \right| + \frac{1}{4}\varepsilon + r^N \frac{1}{4}\varepsilon + r^N \frac{1}{4}\varepsilon \\
&\le \left| \sum_{1\le n}^{<N} (1-r^n)a_n \right| + \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon \\
&= \left| \sum_{1\le n}^{<N} (1-r^n)a_n \right| + \frac{3}{4}\varepsilon
\end{aligned}$$

Whenever $r$ is close enough to 1 that $(1 - r^N) < \frac{\frac{1}{4}\varepsilon}{M_N(N-1)}$, we continue

$$\begin{aligned}
\left|\sum_{1\le n}^{<N}(1-r^n)a_n\right| + \frac{3}{4}\varepsilon &\le \sum_{1\le n}^{<N}|1-r^n|\,|a_n| + \frac{3}{4}\varepsilon \\
&\le \sum_{1\le n}^{<N}\left|1-r^N\right| M_N + \frac{3}{4}\varepsilon \\
&= (N-1)\left|1-r^N\right| M_N + \frac{3}{4}\varepsilon \\
&< \frac{1}{4}\varepsilon + \frac{3}{4}\varepsilon \\
&= \varepsilon
\end{aligned}$$

We have proven that for any $\varepsilon > 0$, if $r < 1$ is sufficiently close to 1, then

$$\left|\sum_{1\le n}^{<\infty} a_n - \sum_{1\le n}^{<\infty} r^n a_n\right| < \varepsilon$$

QED.

**16.**

**(a)** 1

Magnitude of ratio of coefficients:

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\left(\frac{\log(n+1)}{\log n}\right)^2 \\ &= 1^2 \\ &= 1\end{aligned}$$

**(c)** 4

The limit of the magnitude of ratio of coefficients is

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \left(\frac{n+1}{n}\right)^2 \frac{4^n+3n}{4^{n+1}+3(n+1)} \\ &= \left(\frac{n+1}{n}\right)^2 \left(\frac{4^n+\frac{3}{4}(n+1)}{4^{n+1}+3(n+1)} - \frac{\frac{3}{4}(n+1)}{4^{n+1}+3(n+1)} + \frac{3n}{4^{n+1}+3(n+1)}\right) \\ &\to 1(\frac{1}{4}-0+0) \\ &= \frac{1}{4}\end{aligned}$$

**(e)**

**If** $\alpha$ **or** $\beta$ **is in** $\mathbb{Z}_{\leq 0}$**:** r.o.c. is $+\infty$

In this case there are finitely many nonzero terms, so the sum converges everywhere.

**Otherwise:** r.o.c. is 0

Magnitude of ratio of coefficients approaches

$$\begin{aligned}\left|\frac{\alpha+n}{\gamma+n}(\beta+n)\right| &= \left|\left(\frac{\gamma+n}{\gamma+n}+\frac{\alpha-\gamma}{\gamma+n}\right)\right| |\beta+n| \\ &\to (1+0)\,\infty \\ &= \infty\end{aligned}$$