Preamble.

Remark.

Gamelin doesn't specify the domains of functions;
we often know that a function is analytic on $\mathbb{D}$ or $\overline{\mathbb{D}}$,
but such functions could have larger (even disconnected) domains.
I have endeavored to be careful about this in some places.

Blaschke factors.

Throughout this document,

$$B_a(z) = \frac{a - z}{1 - \overline{a}z}$$

for any $a, z \in \mathbb{C}$ such that $1 - \overline{a}z \neq 0$.
Recall the facts about B_a proven in hw1.
Observe also that B_a has a zero of order 1 at a, and no other zeros.

Preliminary lemma. *Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ analytically. Let $f(z_0) = 0$ with multiplicity m. Let*

$$g = \frac{f}{B_{z_0}}$$

Then
(i) g is analytic on $\mathbb{D}$
(ii) $|g| \leq 1$ on $\mathbb{D}$; equality is attained somewhere in $\mathbb{D}$ iff g is constantly some $\lambda \in \partial\mathbb{D}$ iff equality is attained throughout $\mathbb{D}$.
(iii) z_0 has multiplicity $m-1$ as a zero of g.
Note that z_0 need not be a zero of g, since $m-1$ could be 0.
Proof of lemma.

Write

$$f(z) = (z - z_0)^m F(z)$$

$$B_{z_0}(z) = (z - z_0)G(z)$$

Observe:

- F, G are analytic on $\mathbb{D}$.
- F is nonvanishing at z_0.
- G is nonvanishing on $\mathbb{D}$.

Now

$$g(z) = \frac{f}{B_{z_0}}(z) = (z - z_0)^{m-1}\frac{F(z)}{G(z)}$$

which proves *(i)* and *(iii)*.

For *(ii)*,
first observe that $|f| < 1$
since $|f| \leq 1$, f has a zero, and f satisfies the maximum modulus principle.
Second, let $h = f \circ B_{z_0}$ and observe that $h : \mathbb{D} \to \mathbb{D}$ is holomorphic.
Observe also that $h(0) = 0$.
For any $z \in \mathbb{D} \setminus \{z_0\}$, letting $w = B_{z_0}(z)$, we have by the Schwarz lemma:

$$|h(w)| \leq |w|$$

$$\left|\frac{h(w)}{w}\right| \leq 1$$

$$\left|\frac{f(z)}{B_{z_0}(z)}\right| \leq 1$$

Hence $|g| \leq 1$ on $\mathbb{D} \setminus \{z_0\}$, hence on $\mathbb{D}$ by continuity,
hence *(ii)* by maximum modulus principle.
QED.

IX.1 2.

See **Preamble** and **Preliminary lemma**.

If $|z_0| \geq 1$ then the proof is trivial.

Otherwise, applying the lemma m times, we find that the function

$$g(z) = f(z)/B_{z_0}(z)^m$$

is holomorphic on $\mathbb{D}$ with magnitude ≤ 1 there.

Hence

$$\begin{aligned} |f(0)| &= |g(0)|\,|B_{z_0}(0)|^m \\ &= |g(0)|\,|z_0|^m \\ &\leq |z_0|^m \end{aligned}$$

IX.1 3.

f has a zero in $\mathbb{D}$.

Suppose not.
Then $1/f$ is analytic on $\overline{\mathbb{D}}$, hence

$$1 = \left|\frac{1}{f}(0)\right| \leq \max_{\overline{\mathbb{D}}} \left|\frac{1}{f}\right| = \max_{\partial\mathbb{D}} \left|\frac{1}{f}\right| < 1$$

Contradiction, QED.

f has no zeros in $D_{1/M}(0)$.

Let $z_0 \in \mathbb{D}$ be a root of f; let m be its multiplicity. Let

$$M' := \max_{\overline{\mathbb{D}}} |f| = \max_{\partial\mathbb{D}} |f| < M$$

z_0 is a root of f/M' with multiplicity m
and $f/M' : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ analytically,
hence the previous exercise gives

$$1/M < 1/M' = \left|\frac{f}{M'}(0)\right| \leq |z_0|^m \leq |z_0|$$

QED.

IX.2 2.

$$
\begin{aligned}
\frac{1+3z^2}{3+z^2} &= \frac{\frac{1}{3}+z^2}{1+\frac{1}{3}z^2} \\
&= \frac{(\frac{1}{\sqrt{3}}-iz)}{(1-\frac{1}{\sqrt{3}}iz)}\frac{(\frac{1}{\sqrt{3}}+iz)}{(1+\frac{1}{\sqrt{3}}iz)} \\
&= \frac{(-i\frac{1}{\sqrt{3}}-z)}{(1-\frac{1}{\sqrt{3}}iz)}\frac{(i\frac{1}{\sqrt{3}}-z)}{(1+\frac{1}{\sqrt{3}}iz)} \\
&= \frac{(-i\frac{1}{\sqrt{3}}-z)}{(1-\overline{-i\frac{1}{\sqrt{3}}}z)}\frac{(i\frac{1}{\sqrt{3}}-z)}{(1-\overline{i\frac{1}{\sqrt{3}}}z)}
\end{aligned}
$$

IX.2 3.

See **Preamble** and **Preliminary lemma**.
Applying the lemma four times to the function

$$g = f \circ (z \mapsto 3z)$$

we have

$$\left|\frac{g}{B_{1/3} \cdot B_{i/3} \cdot B_{-1/3} \cdot B_{-i/3}}\right| \leq 1$$

$$|f(0)| = |g(0)| \leq \left|\left(B_{1/3} \cdot B_{i/3} \cdot B_{-1/3} \cdot B_{-i/3}\right)(0)\right| \; = \frac{1}{81}$$

with equality
iff $g/\left(B_{1/3} \cdot B_{i/3} \cdot B_{-1/3} \cdot B_{-i/3}\right)$ is constantly (on $\mathbb{D}$) some $\lambda \in \partial\mathbb{D}$
iff $g = \lambda \cdot \left(B_{1/3} \cdot B_{i/3} \cdot B_{-1/3} \cdot B_{-i/3}\right)$ on $\mathbb{D}$
iff

$$f = \lambda \cdot \left(B_{1/3} \cdot B_{i/3} \cdot B_{-1/3} \cdot B_{-i/3}\right) \circ (z \mapsto \tfrac{1}{3}z)$$

on $D_3(0)$.
Hence those f which satisfy the above equation on $D_3(0)$ for some $\lambda \in \partial\mathbb{D}$
are precisely those which attain $|f(0)| = 1/81$
and $1/81$ is the maximum.

IX.2 4.

See the **Blaschke factors** section of **Preamble**.

Answer.

If $z_0 = z_1$, then the maximum value is 0 and every f attains it.

Otherwise, we will show that the maximum is

$$2 - \frac{4}{1 + e^{\rho(z_0, z_1)/2}}$$

(where ρ is hyperbolic distance),
and that the $f \in \mathrm{End}(\mathbb{D})$ which attain it
are precisely those which lie in $\mathrm{Aut}(\mathbb{D})$ and satisfy $f(z_0) + f(z_1) = 0$;
further, we will show that these f
are precisely the functions $\lambda B_{\mathrm{h.mp.}(z_0, z_1)}$ for $\lambda \in \partial\mathbb{D}$,
where h. mp. denotes hyperbolic midpoint.

Definition. Let $z_0, z_1 \in \mathbb{D}$. A **hyperbolic midpoint of** z_0, z_1 is a point $a \in \mathbb{D}$ such that

$$\rho(z_0, a) = \rho(a, z_1) = \tfrac{1}{2}\rho(z_0, z_1)$$

Theorem 1. *Every pair of points $z_0, z_1 \in \mathbb{D}$ has a unique hyperbolic midpoint; it lies on the geodesic from z_0 to z_1.*

Proof.

Let $\gamma : [0, 1] \to \mathbb{D}$ be continuously differentiable and parametrize the geodesic from z_0 to z_1,
with $\gamma(0) = z_0$ and $\gamma(1) = z_1$.
(In particular, the length of γ is $\rho(z_0, z_1)$.)
Observe that $\forall t \in [0, 1]$:
$\gamma|_{[0,t]}$ parametrizes the geodesic from z_0 to $\gamma(t)$ and
$\gamma|_{[t,1]}$ parametrizes the geodesic from $\gamma(t)$ to z_1.
Hence:

$$\begin{aligned}
\rho(z_0, z_1) &= 2\int_0^1 \frac{|\gamma'(s)|}{1 - |\gamma(s)|^2} ds \\
&= 2\int_0^t \frac{|\gamma'(s)|}{1 - |\gamma(s)|^2} ds + 2\int_t^1 \frac{|\gamma'(s)|}{1 - |\gamma(s)|^2} ds \\
&= \rho(z_0, \gamma(t)) + \rho(\gamma(t), z_1)
\end{aligned}$$

Clearly $t \mapsto \rho(z_0, \gamma(t))$ is continuous in t,
so the intermediate value theorem proves that for some t,
$a := \gamma(t)$ is a hyperbolic midpoint of z_0, z_1.
This proves existence.

We now prove uniqueness among points on the geodesic.
If $t_0 < t_1 \in [0, 1]$ with $\gamma(t_0) \neq \gamma(t_1)$, then

$$\begin{aligned}
\rho(z_0, \gamma(t_1)) - \rho(z_0, \gamma(t_0)) &= 2\int_{t_0}^{t_1} \frac{|\gamma'(s)|}{1 - |\gamma(s)|^2} ds \\
&\neq 0
\end{aligned}$$

so $\rho(z_0, \gamma(t_0)) \neq \rho(z_0, \gamma(t_0))$,
so a is the only point on the geodesic satisfying $\rho(z_0, a) = \frac{1}{2}\rho(z_0, z_1)$.

Now if b is a hyperbolic midpoint of z_0, z_1,
we have $\rho(z_0, b) + \rho(b, z_1) = \rho(z_0, z_1)$.
Letting γ_0 be the geodesic from z_0 to b and γ_1 the geodesic from b to z_1,
we find that the concatenation (joining) of γ_0, γ_1 is a path of shortest length from z_0 to z_1,
i.e. it is their geodesic,
hence $b = a$ by the uniqueness of the hyperbolic midpoint on the geodesic.
This proves uniqueness in general, and completes the theorem.

Theorem 2. *Let* $z_0, z_1 \in \mathbb{D}$. *Then* $\mathrm{h.\,mp.}(z_0, z_1) = 0 \iff z_0 + z_1 = 0$.
Proof.

Gamelin showed that hyperbolic geodesics passing through 0 are straight lines.
Hence for any $z \in \mathbb{D}$: $\gamma(t) = tz$ is the geodesic from 0 to z and

$$\begin{aligned}\rho(z, 0) &= 2\int_0^1 \frac{|\gamma'(t)|dt}{1 - |\gamma(t)|^2} \\ &= 2\int_0^1 \frac{|z|dt}{1 - t^2|z|^2} \\ &= 2\int_0^{|z|} \frac{dr}{1 - r^2}\end{aligned}$$

This is strictly increasing as a function of $|z|$,
so $\rho(0, z)$ is *uniquely* determined by $|z|$ (and vice versa).

$\Longleftarrow$:

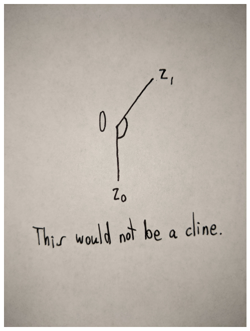
Suppose $z_0 + z_1 = 0$.
Then the straight line between them
is a cline orthogonal to $\partial\mathbb{D}$
hence is their geodesic.
It passes through 0, so

$$\rho(z_0, 0) + \rho(0, z_1) = \rho(z_0, z_1)$$

(We showed in the proof of the previous theorem
that this equation holds when 0 is replaced
with any point on the hyperbolic shortest path from z_0 to z_1.)
We also have $|z_0| = |z_1|$ by hypothesis
hence $\rho(z_0, 0) = \rho(z_1, 0) = \frac{1}{2}\rho(z_0, z_1)$
hence $\mathrm{h.\,mp.}(z_0, z_1) = 0$.

$\Longrightarrow$:

Suppose $\mathrm{h.\,mp.}(z_0, z_1) = 0$.
Assume $z_0 \neq z_1$ (otherwise trivial).
Then $|\mathbf{z_0}| = |\mathbf{z_1}|$**.**
Let γ_0, γ_1 be the geodesics from z_0 to 0 and 0 to z_1 respectively.
The end of γ_0 meets the start of γ_1 at 0, with an angle of $\arg z_1 - \arg z_0$.
The concatenation (joining) of γ_0, γ_1 forms this angle at 0,
hence it can only be a cline if the angle is straight (180°);
indeed, if the angle was not straight, we would have



The concatenation has length $\rho(z_0, 0) + \rho(0, z_1) = \rho(z_0, z_1)$
(which is the shortest length of any path from z_0 to z_1),
so it is a geodesic, hence a cline, hence the angle is straight,
meaning that $\mathbf{z_0}, \mathbf{z_1}$ **have opposite direction.**
Since z_0, z_1 have same magnitude and opposite direction,
we conclude that $z_0 + z_1 = 0$.

Observation.

If $f \in \mathrm{Aut}(\mathbb{D})$,
then for any $z_0, z_1 \in \mathrm{Aut}(\mathbb{D})$:

$$f(\mathrm{h.\,mp.}(z_0, z_1)) = \mathrm{h.\,mp.}(f(z_0), f(z_1))$$

because f is an isometry with respect to ρ.

Theorem 3. *Let $z_0, z_1 \in \mathbb{D}$, $a = \mathrm{h.\,mp.}(z_0, z_1)$, $R = |B_a(z_0)| = |B_a(z_1)|$. Then*

$$|z_0 - z_1| \leq 2R$$

with equality iff $a = 0$.
(Note that R is well-defined because $\mathrm{h.\,mp.}(B_a(z_0), B_a(z_1)) = B_a(a) = 0$.)
Proof.

Let $\lambda = B_a(z_0)/R$, so that $z_0 = \lambda R$ and $z_1 = -\lambda R$.
Observe that, since $0 \leq R < 1$:

$$\frac{1 - |a|^2}{|1 - (\overline{a}\lambda R)^2|} \leq 1$$

with equality iff $a = 0$.

Now

$$
\begin{aligned}
|z_0 - z_1| &= |B_a(\lambda R) - B_a(-\lambda R)| \\
&= \left| \frac{a - \lambda R}{1 - \overline{a}\lambda R} - \frac{a + \lambda R}{1 + \overline{a}\lambda R} \right| \\
&= \left| \frac{(a - \lambda R)(1 + \overline{a}\lambda R) - (a + \lambda R)(1 - \overline{a}\lambda R)}{1 - (\overline{a}\lambda R)^2} \right| \\
&= \left| \frac{(a - \lambda R + |a|^2\lambda R - \overline{a}(\lambda R)^2) - (a + \lambda R - |a|^2\lambda R - \overline{a}(\lambda R)^2)}{1 - (\overline{a}\lambda R)^2} \right| \\
&= \left| \frac{-2\lambda R + 2|a|^2\lambda R}{1 - (\overline{a}\lambda R)^2} \right| \\
&= 2R \frac{1 - |a|^2}{|1 - (\overline{a}\lambda R)^2|} \\
&\leq 2R
\end{aligned}
$$

with equality iff $a = 0$.

Theorem 4. *Let z_0, z_1, a, R be as in the previous theorem; let $\rho = \rho(z_0, z_1)$. Then*

$$2R = 2 - \frac{4}{1 + e^{\rho/2}}$$

Proof.

$$
\begin{aligned}
\rho &= 2 \int_{-R}^{R} \frac{dr}{1 - r^2} \\
&= 2 \int_{-R}^{R} \frac{1}{2} \left(\frac{1}{1 + r} + \frac{1}{1 - r} \right) dr \\
&= 2\tfrac{1}{2} \big[\log(1 + r) - \log(1 - r) \big]_{r=-R}^{R} \\
&= \log \frac{1 + R}{1 - R} - \log \frac{1 - R}{1 + R} \\
&= 2 \log \frac{1 + R}{1 - R} \\
&= 2 \log \left(-1 + \frac{2}{1 - R} \right)
\end{aligned}
$$

$$
\begin{gathered}
\rho = 2 \log \left(-1 + \frac{2}{1 - R} \right) \\
1 + e^{\rho/2} = \frac{2}{1 - R} \\
\frac{2}{1 + e^{\rho/2}} = 1 - R \\
\frac{2}{1 + e^{\rho/2}} - 1 = -R
\end{gathered}
$$

$$R = 1 - \frac{2}{1 + e^{\rho/2}}$$

$$2R = 2 - \frac{4}{1 + e^{\rho/2}}$$

Note.

Recall from Gamelin that for any $f \in \mathrm{End}(\mathbb{D})$ and any distinct $z_0, z_1 \in \mathbb{D}$:

$$\rho(f(z_0), f(z_1)) \leq \rho(z_0, z_1)$$

with equality iff $f \in \mathrm{Aut}(\mathbb{D})$.

Conclusion.

Let $f \in \mathrm{End}(\mathbb{D})$ and $z_0, z_1 \in \mathbb{D}$, with $z_0 \neq z_1$.
Let $a = \mathrm{h.\,mp.}(z_0, z_1)$.
Then

$$|f(z_0) - f(z_1)| \ \leq 2 - \frac{4}{1 + e^{\rho(f(z_0), f(z_1))/2}} \ \leq 2 - \frac{4}{1 + e^{\rho(z_0, z_1)/2}}$$

with equality in the first iff $f(z_0) + f(z_1) = 0$,
and equality in the second iff $\rho(f(z_0), f(z_1)) = \rho(z_0, z_1)$ iff $f \in \mathrm{Aut}(\mathbb{D})$.

If $f \in \mathrm{Aut}(\mathbb{D})$,
then equality holds in the first iff $\mathrm{h.\,mp.}(f(z_0), f(z_1)) = 0$
iff $f(a) = 0$
iff $f = \lambda B_a$ for some $\lambda \in \partial\mathbb{D}$.

Hence

$$|f(z_0) - f(z_1)| \leq 2 - \frac{4}{1 + e^{\rho(z_0, z_1)/2}}$$

with equality iff $f \in \mathrm{Aut}(\mathbb{D})$ and $f(z_0) + f(z_1) = 0$
iff $f = \lambda B_a$ for some $\lambda \in \partial\mathbb{D}$.