# Math 185: Homework 2 Solution

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The following exercises are from Stein's textbook, Chapter 1.

## 1 Problem 1

**Problem** (1.10)**.** *Show that*

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \Delta$$

*where* $\Delta$ *is the* **Laplacian***.*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

**Solution.** From page 12 of Stein, we get

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Plug in and you get the desired result.

## 2 Problem 2

**Problem** (1.11)**.** *Use Exercise 10 to show that if $f$ is holomorphic, then the real part and imaginary part of $f$ is harmonic.*

**Solution.** If $f$ is holomorphic, then $\partial f/\partial \overline{z} = 0$ everywhere, hence further derivative

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f = 0$$

as well. The same argument can be applied to function $\overline{f}$, which sends $z$ to $\overline{f(z)}$. More precisely, we have

$$\frac{\partial}{\partial z}\overline{f} = \overline{\left(\frac{\partial f}{\partial \overline{z}}\right)} = 0.$$

then apply $\partial/\partial barz$ to it, we get

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\overline{f} = 0.$$

Thus

$$\Delta \mathrm{Re} f = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} \frac{f + \overline{f}}{2} = 0$$

The case for imaginary part is similar.

Alternatively, you can write $f = u + iv$ for the real and imaginary part, then use Cauchy Riemann condition to get

$$\partial_x^2 u + \partial_y^2 u = \partial_x(\partial_y v) + \partial_y(-\partial_x v) = 0.$$

**Remark.** Hmm, every holomorphic function has its real part being a harmonic function. Does every harmonic arise in this way? Namely, given a harmonic function $u$, can we find another harmonic function $v$, such that $f = u + iv$ is a holomorphic function?

# 3 Problem 3

**Problem** (1.13)**.** *Suppose that $f$ is holomorphic in an open set $\Omega$. Prove that in any one of the following cases:*

1. *Re($f$) is constant.*
2. *Im($f$) is constant.*
3. *$|f|$ is constant.*

*One can conclude $f$ is constant.*

**Solution.** Let $f = u + iv$.

If $u$ is constant, then by Cauchy Riemann condition, we know $\partial_x v = -\partial_y u = 0$ and $\partial_y v = \partial_x u = 0$, hence $v$ is constant. Thus $f$ is constant.

If $v$ is constant, by similar argument, we know $u$ is constant.

If $|f|$ is constant and non-zero, we can say $\mathrm{Re}(\log f)$ is constant, hence $\log f$ is constant. If you complain that we have not learned log, then we can look at problem 9. If you complain that we haven't done problem 9, then we can consider $|f|^2 = u^2 + v^2$ being constant, then

$$0 = \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} (f \overline{f}) = \frac{\partial f}{\partial z} \frac{\partial \overline{f}}{\partial \overline{z}} = \left| \frac{\partial f}{\partial z} \right|^2 \Rightarrow \frac{\partial f}{\partial z} = 0.$$

This forces $f$ being a constant, thanks to page 23 Corollary 3.4.

# 4 Problem 4

**Problem** (Ex 1.15)**.** *If $\sum_{n=1}^{\infty} a_n$ converges, show that*

$$\lim_{r \to 1^-} \sum_{n=1}^{\infty} r^n a_n = \lim_{r \to 1-} \sum_{n=1}^{\infty} a_n.$$

**Solution.** Let $A_N = \sum_{n=1}^N a_n$, with $A_0 = 0$, then

$$C(N,r) = \sum_{n=1}^N r^n a_n = \sum_{n=1}^N r^n (A_n - A_{n-1}) = \sum_{n=1}^N r^n A_n - \sum_{n=1}^{N-1} r^{n+1} A_n = \sum_{n=1}^{N-1} (1-r) r^n A_n + r^N A_N.$$

We want to show that

$$\lim_{r \to 1^-} \lim_{N \to \infty} C(N,r) = A$$

note that the ordre of the limit cannot be changed without justification. We note that $A = \lim_n A_n = \sum_n a_n$ exists, hence

$$\lim_{N\to\infty} r^N A_N = \lim_{N\to\infty} r^N \lim_N A_N = 0 \cdot A = 0.$$

Next, we consider

$$\sum_{n=1}^{\infty} (1-r) r^n A_n = \sum_{n=1}^{\infty} (1-r) r^n (A + (A_n - A)) = A \sum_n (1-r) r^n + \sum_n (1-r) r^n B_n = A + \sum_n (1-r) r^n B_n.$$

where we let $B_n = A_n - A$, thus $\lim B_n = 0$. We need to show that

$$\lim_{r\to 1^-} \sum_n (1-r) r^n B_n = 0.$$

For any $\epsilon > 0$, it suffice to show that

$$\lim_{r\to 1^-} |\sum_n (1-r) r^n B_n| < \epsilon.$$

Since $B_n \to 0$, we know that there exists $N > 0$, such that for any $n \geq N$, $|B_n| < \epsilon$. Thus, we have

$$|\sum_{n=N}^{\infty} (1-r) r^n B_n| \leq \sum_{n=N}^{\infty} (1-r)^{rn} \epsilon = \epsilon r^N \sum_{n=0}^{\infty} (1-r) r^n = r^N \epsilon < \epsilon.$$

Thus

$$\begin{aligned}
\lim_{r\to 1^-} |\sum_n (1-r) r^n B_n| &\leq \lim_{r\to 1^-} |\sum_{n=1}^{N-1} (1-r) r^n B_n| + |\sum_{n=N}^{\infty} (1-r) r^n B_n| \\
&< \lim_{r\to 1} \sum_{n=1}^{N-1} (1-r) r^n |B_n| + \epsilon \\
&= \sum_{n=1}^{N-1} 0 |B_n| + \epsilon = \epsilon
\end{aligned}$$

where in the next to last step, we can switch the order of the **finite** summation $\sum_{n=1}^{N-1}$ and $\lim_{r\to 1}$. Thus we have finished the proof.

## 5 Problem 5

**Problem.** *Exercise 16 (a) (c) (e) Determine the radius of convergence of the series* $\sum_{n=1}^{\infty} a_n z^n$ *when*

*(a)* $a_n = (\log n)^2$

*(c)* $a_n = \frac{n^2}{4^n+3n}$

*(e) Find the radius of convergence for the hypergeometric series*

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n$$

*Here* $\alpha, \beta \in \mathbb{C}$ *and* $\gamma \neq 0, -1, -2, \cdots$.

**Solution.** (a) We have limit

$$(1/n)\log|a_n| = (2/n)\log|\log n| \to 0 \quad \text{as } n \to \infty$$

Hence $1/R = e^0 = 1$, and $R = 1$.

(c) We have limit (better than lim sup)

$$1/R = \lim_{n\to\infty} \left|\frac{n^2}{4^n + 3n}\right|^{1/n} = \frac{\lim_{n\to\infty} |n|^{2/n}}{4\lim(1+3n4^{-n})^{1/n}} = \frac{1}{4}$$

where we used the rules that, if $a = \lim_n a_n, b = \lim_n b_n$, then $\lim_n a_n b_n = ab, \lim_n a_n^{b_n} = a^b$ etc. Hence $R = 4$.

(e) Using ratio test (as justified in exercise 17), we have

$$\frac{a_n}{a_{n-1}} = \frac{(\alpha+n-1)(\beta+n-1)}{n(\gamma+n-1)} = \frac{(1+\frac{\alpha-1}{n})(1+\frac{\beta-1}{n})}{(1+\frac{\gamma-1}{n})} \to 1 \quad \text{as } n \to \infty$$

Hence $R = 1$.