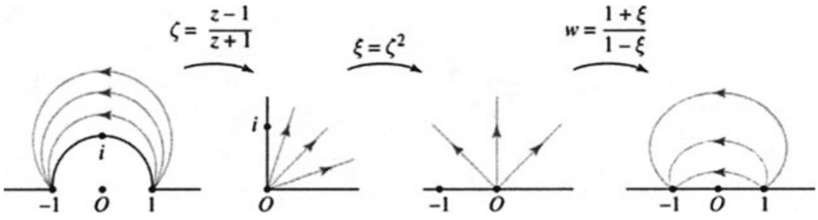
found to be $w = (1+\xi)/(1-\xi)$. The final solution is then

$$(1.1)\quad w = \frac{1+(z-1)^2/(z+1)^2}{1-(z-1)^2/(z+1)^2} = \frac{(z+1)^2+(z-1)^2}{(z+1)^2-(z-1)^2} = \frac{1}{2}\left(z+\frac{1}{z}\right),$$

which has already appeared in Exercise II.6.6.



Exercises for XI.1

1. Find a conformal map of the sector $\{|\arg z| < \pi/3\}$ onto the open unit disk mapping 0 to -1 and ∞ to $+1$. Sketch the images of radial lines and of arcs of circles centered at 0. Is the map unique?

2. Find a conformal map of the slit plane $\mathbb{C}\backslash(-\infty, 0]$ onto the open unit disk satisfying $w(0) = i$, $w(-1+0i) = +1$, $w(-1-0i) = -1$. What are the images of circles centered at 0 under the map? Sketch them.

3. For fixed $A > 0$, find the conformal map $w(z)$ of the open unit disk $\{|z| < 1\}$ onto the vertical strip $\{-A < \operatorname{Re} w < A\}$ that satisfies $w(0) = 0$ and $w'(0) > 0$. Sketch the curves in the disk that correspond to vertical and horizontal lines in the strip.

4. Find a conformal map $w(z)$ of the strip $\{\operatorname{Im} z < \operatorname{Re} z < \operatorname{Im} z + 2\}$ onto the upper half-plane such that $w(0) = 0$, $w(z) \to +1$ as $\operatorname{Re} z \to -\infty$, and $w(z) \to -1$ as $\operatorname{Re} z \to +\infty$. Sketch the images of the straight lines $\{\operatorname{Re} z = \operatorname{Im} z + c\}$ in the strip. What is the image of the median line $\{\operatorname{Re} z = \operatorname{Im} z + 1\}$ of the strip?

5. Find a conformal map $w(z)$ of the right half-disk $\{\operatorname{Re} z > 0,\ |z| < 1\}$ onto the upper half-plane that maps $-i$ to 0, $+i$ to ∞, and 0 to -1. What is $w(1)$?

6. Let $w = g(z)$ be the conformal map of the right half-disk $\{\operatorname{Re} z > 0,\ |z| < 1\}$ onto the entire unit disk that fixes the points $\pm i$ and $+1$. (a) Without computing $g(z)$ explicitly, show that $g(\bar{z}) = \overline{g(z)}$. *Hint.* Argue that $h(z) = \overline{g(\bar{z})}$ is another conformal map satisfying the same conditions, and appeal to uniqueness. (b) Use symmetry to show that $g(0) = -1$. (In other words, use part (a).) (c) Find

$g(z)$ as a composition of explicit conformal maps, and use this to check that $g(0) = -1$.

7. Find the conformal map of the pie-slice domain $\{|\arg z| < \alpha, |z| < 1\}$ onto the open unit disk such that $w(0) = -1$, $w(+1) = +1$, and $w(e^{i\alpha}) = i$. It is enough to express $w(z)$ as a composition of specific conformal maps.

8. For fixed b in the interval $(-1, 1)$, find all conformal maps of the unit disk slit along the interval $[-1, b]$ onto the entire unit disk that map b to -1 and leave $+1$ fixed. It is enough to express them as a composition of specific conformal maps.

9. For fixed $a > 0$, let D be the domain obtained by slitting the upper half-plane along the vertical interval from $z = 0$ to $z = ia$.
   (a) Find a conformal map $w(z)$ of D onto the entire upper half-plane such that $w(z) \sim z$ as $|z| \to \infty$. *Hint.* Consider the preliminary map $\zeta = z^2$.
   (b) Describe how the map can be used to model the flow of water over a vertical metal sheet lying in a flat river bed, perpendicular to the flow of the water. Give a rough sketch of the streamlines of the flow.

10. Find the conformal map $w = f(z)$ of the exterior domain $\{|z| > 1\} \cup \{\infty\}$ onto $\mathbb{C}^* \setminus [-1, +1]$ such that $f(\infty) = \infty$ and the argument of $f(z)/z$ tends to α as $z \to \infty$, where α is a fixed real number. Sketch the images of circles $\{|z| = r\}$, for $r > 1$, and the images of the intervals $(-\infty, -1]$ and $[1, \infty)$. What is the inverse map? (Specify the branch.) *Hint.* For $\alpha = 0$, use the map $(z + 1/z)/2$ treated in (1.1) above. For the general case, do a preliminary rotation. See also the exercises in Section II.6.

11. Show that the half-strip $\{-\pi/2 < \operatorname{Re} z < \pi/2, \operatorname{Im} z > 0\}$ is mapped conformally by $w = \sin z$ onto the upper half-plane. Sketch the images of horizontal and vertical lines.

12. Find a conformal map of the half-strip in Exercise 11 onto the open unit disk that maps $-\pi/2$ to $-i$, $\pi/2$ to i, and 0 to $+1$. Where does ∞ go under this map?

2. The Riemann Mapping Theorem

This section is devoted to a preliminary discussion of the Riemann mapping theorem. The proof will be postponed to the end of the chapter. The Riemann mapping theorem was first established in the following generality by W.F. Osgood in 1900.