Lecture 6: 02/03/22

Lebesgue criterion for measurability

$E \subseteq \mathbb{R}^n$ measurable if $\exists$ a G_δ-set G and F_σ-set F s.t. $G \supseteq E \supseteq F$ and $m(G \setminus F) = 0$.

G is a hull of E, F a kernel, these are unique up to a null set

Lemma: If $G_1, G_2, \ldots$ are G_δ-sets, $\bigcap G_i$ is a G_δ set.

Pf: $G_1 = \bigcap_{j=1}^{\infty} U_{1,j}$, $G_2 = \bigcap_{j=1}^{\infty} U_{2,j}$ (U_i open)

$G = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} U_{ij}$ still intersection of countably many open G_δ-set

Lemma: $F_1 \ldots F_2$ F_σ set, then $\bigcup_{i=1}^{\infty} F_i$ is F_σ

Thm: (Product + Slices): If $E \subseteq \mathbb{R}^n$, $F \subseteq \mathbb{R}^k$ measurable, then $m(E \times F) = m(E)\, m(F)$ and if $m(E) = 0$ or $m(F) = 0$, $m(E \times F) = 0$.

Proof:

$(n=1, k=1)$

Lemma: If $E \subseteq \mathbb{R}$ has $m(E) = 0$, then $m(E \times \mathbb{R}) = 0$

Pf: $\forall n > 0$ int, we will cover E by collection of boxes w/ total area $\frac{\varepsilon}{2^{2n+1}}$, $\{B_{n,i}\}_{i=1}^{\infty}$.

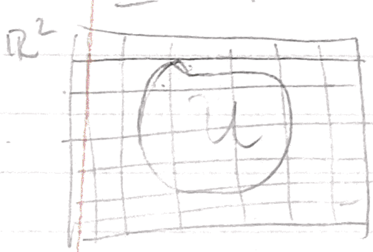
Then define $\tilde{B}_{n,i} = B_{n,i} \times (-2^n, 2^n)$, $\sum_i |\tilde{B}_{n,i}| \leq \frac{\varepsilon}{2^n}$. Then $\sum_{n,i} |\tilde{B}_{n,i}| \leq \varepsilon$.

Thus since $E \times F \subseteq E \times \mathbb{R}$, $m(E \times F) = 0$.

<u>Lemma</u>: $m(E \times F) = m(E) \times m(F)$ for
- E, F open boxes
- E, F are boxes
- E, F open?

<u>Lemma</u>: Any open set $\mathcal{U} \subseteq \mathbb{R}^n$ can be written as a countable disjoint union of open boxes and a measure zero set.

<u>Pf</u>: (take $n=2$) Consider unit open squares in $\mathbb{R}^2$ $(a, a+1) \times (b, b+1)$ $a, b \in \mathbb{Z}$.



For size 1 open boxes in $\mathcal{U}$, we take them.

For boxes disjoint from $\mathcal{U}$, ignore.

For open box B s.t. $B \cap \mathcal{U} \neq \emptyset$, we further divide into 4 size 1/2 pieces.

$\forall x \in \mathcal{U}$, it is either covered by some boxes or contained in a boundary of certain box.

Union of boundaries of boxes is a null set

Then, for E, F open we can write

$$E = \bigsqcup_{i=1}^{\infty} B_i + Z$$
$$F = \bigsqcup_{i=1}^{\infty} B_i' + Z'$$

Then
$$m(E \times F) = m\left(\left(Z + \bigsqcup_{i=1}^{\infty} B_i\right) \times \left(Z' + \bigsqcup B_i'\right)\right)$$
$$= m\left(\left(\bigsqcup_{i=1}^{\infty} B_i\right) \times \left(\bigsqcup_{j=1}^{\infty} B_j'\right)\right)$$
$$= \sum_{i,j} m(B_i) \times m(B_j')$$
$$= \left(\sum_i m(B_i)\right) \times \left(\sum_j m(B_j')\right)$$
$$= m(E) \times m(F)$$

Lemma: For E, F measurable subsets of $\mathbb{R}$ now prove $m(E \times F) = m(E) \times m(F)$

Pf: Assume E, F bounded. Let $H_E \supset E \supset K_E$, $H_F \supset F \supset K_F$ s.t. $m(H_E \setminus K_E) = 0$, $m(H_F \setminus K_F) = 0$

Since H_E, H_F are G_δ set, $m(H_E \times H_F) = m(H_E) \times m(H_F)$

Claim: • $m(H_E \times H_F \setminus K_E \times K_F) = 0$
• $H_E \times H_F$ is G_δ, $K_E \times K_F$ is F_σ-set

Note $H_E \times H_F \setminus K_E \times K_F \subseteq (H_E \setminus K_E) \times H_F \cup H_E \times (H_F \setminus K_F)$

$\leadsto$ $E \times F$ measurable

$m(E \times F) = m(H_E \times H_F) = m(H_E) \times m(H_F)$
$= m(E) \times m(F)$

Let $E \subseteq \mathbb{R}^n \times \mathbb{R}^k$, For any $x \in \mathbb{R}^n$,
let $E_x = E \cap \{x\} \times \mathbb{R}^k \subseteq \{x\} \times \mathbb{R}^k \cong \mathbb{R}^k$

Thm: Let $Z = \{x \in \mathbb{R}^n \mid m_{\mathbb{R}^k}(E_x) \neq 0\}$ is measure zero in $\mathbb{R}^n$, Then $m(E) = 0$.

pf: Let $\tilde{E} = E \setminus Z \times \mathbb{R}^k$, Then $m(Z \times \mathbb{R}^k) = 0$,
$m(\tilde{E}) = m(E)$. Suffices to show $m(\tilde{E}) = 0$. Replace E by $\tilde{E}$, assume $Z = \emptyset$, i.e. $m(E_x) = 0$ $\forall x$.

Assume E is a bounded subset in $\mathbb{R}^2$, and $E \subset [0,1]^2$, $m(E_x) = 0$ $\forall x$

$\forall \varepsilon > 0$, WTS $m(E) < \varepsilon$.

① find $K \subset E$ closed, s.t. $m(E \setminus K) \leq \varepsilon/2$

Then K is compact, $m(K_x) = 0$ $\forall x$

② Cover K by "boxes" of total area $\leq \varepsilon/2$ $\Rightarrow$ $m(K) \leq \varepsilon/2$

$\forall x \in \mathbb{R}$ if $K_x \neq \emptyset$ we can find $V(x) \subseteq \mathbb{R}$ open s.t. $m(V(x)) < \varepsilon/2$, $V(x) \supset K_x$

<u>Claim:</u> $\exists$ $\mathcal{U}(x) \ni x$ s.t. $\mathcal{U}(x) \times V(x) \supset \pi^{-1}(\mathcal{U}(x)) \cap K$ i.e., $\forall y \in \mathcal{U}(x)$, want $V_x \supset K_y$

Suppose $\forall$ $\varepsilon > 0$ $\exists$ $\tilde{x}$ w/ $|\tilde{x} - x| < \varepsilon$ s.t. $V_x \not\supset K_{\tilde{x}}$, $\exists$ $(\tilde{x}, \tilde{y}) \subseteq K_{\tilde{x}}$ s.t. $\tilde{y} \notin V_x$

Then $\exists$ $(\tilde{x}_n, \tilde{y}_n) \in K$ s.t. $\tilde{x}_n \to x$, $\tilde{y}_n \notin V(x)$. By passing to subseq, we may assume $(\tilde{x}_n, \tilde{y}_n) \to (x, y) \in K_x$

So $\tilde{y}_n \to y$ but $\tilde{y}_n \in V_x^c \Rightarrow y \in V_x^c$.

But this is a contradiction because $y \in K_x \subseteq V_x$.

Thus $\forall x \in \mathbb{R}$, $\exists$ $V_x \supset K_x$ (open), $m(V_x) < \varepsilon/2$

$\exists$ $\mathcal{U}_x \ni x$, s.t. $\mathcal{U}_x \times V_x \supset \pi^{-1}(\mathcal{U}_x) \cap K$

$K \subseteq \bigcup_{x \in \mathbb{R}} V_x \times \mathcal{U}_x$, K compact we can pass to finite subcover $K \subset V_{x_1} \times \mathcal{U}_{x_1} \cup \dots \cup V_{x_N} \times \mathcal{U}_{x_N}$

Take $\mathcal{U}_i = \mathcal{U}_{x_i} \setminus (\mathcal{U}_{x_1} \cup \mathcal{U}_{x_2} \cup \dots \mathcal{U}_{x_{i-1}})$

$\Rightarrow$ $K \subset V_{x_1} \times \mathcal{U}_1 \sqcup V_{x_2} \times \mathcal{U}_2 \sqcup \dots$

$\sum m(V_{x_i} \times \mathcal{U}_i) = \sum m(V_{x_i}) \times m(\mathcal{U}_i) \leq \frac{\varepsilon}{2} \sum m(\mathcal{U}_i) < \frac{\varepsilon}{2}$