Lecture 1 - 01/18/21

- Riemann integral over $\mathbb{R}$

$$\int_a^b f(x)\,dx = \text{area under the curve}$$

At least piecewise cont. fns are Riemann integrable

- Short comings:
  - underlying space is like $\mathbb{R}$, $\mathbb{R}^n$, (domain of int is only bounded $[a,b]$)
  - only bounded functions are considered
  - if $f_n \to f$ pointwise, and f_n is Riemann integrable, f may not be

- Lebesgue integral

$$\int_\Omega f\,dx, \quad \Omega \subseteq \mathbb{R}^n, \quad dx = dx_1 \cdots dx_n$$

  - what Ω allowed? (Lebesgue measurable sets)
  - what f allowed? (Lebesgue integrable fn)

- Lebesgue measure:

$$m(\Omega) = \int_\Omega 1\,dx$$

ex. $\int_a^b 1\,dx = b-a = |[a,b]|$

Intuitively,

1. $\Omega \subset \mathbb{R} \Rightarrow m(\Omega) = \text{length}$
2. $\Omega \subseteq \mathbb{R}^2 \Rightarrow m(\Omega) = \text{area}$
3. $\Omega \subseteq \mathbb{R}^3 \Rightarrow m(\Omega) = \text{volume}$

Desired properties of measure:

1. monotone: $A \subset B \subset \mathbb{R}^n \Rightarrow m(A) \leq m(B)$
2. additivity: If $A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) + m(B)$
3. translation invariance: $\forall x \in \mathbb{R}^n$, $E \in \mathbb{R}^n$
   $m(E) = m(x + E)$

where

$$x + E = \{x + a \mid a \in E\}$$

Problem: <u>not possible</u> on <u>all</u> subsets on $\mathbb{R}^n$

- Read Tao 7.3
- Read wiki: Banach-Tarski Paradox
  - unit ball in $\mathbb{R}^3$ = disjoint union of finitely many pieces w/ after rotation, translations, one can make 2 unit balls

Cure: Restrict class of subsets in $\mathbb{R}^n$ to which we assign a measure, which we call "measurable"

<u>Desired Props / Axioms</u> of measurable subsets:

Let M_n denote set of measurable subsets in $\mathbb{R}^n$.

notation: 2^S = set of subsets of S

We want

1. if $U \subset \mathbb{R}^n$ is open, $U \in M_n$
2. if $U \in M_n$, then $U^c = \mathbb{R}^n \setminus U \in M_n$
3. if $U, V \in M_n$, $U \cap V$, $U \cup V \in M_n$

②, ③ implies M_n is a Boolean Algebra, a set w/ 3 operations NOT, AND, OR
$(\,)^c$ $(\,)\cap(\,)$ $(\,)\cup(\,)$

④ We want M_n to be a σ-algebra, i.e. if $U_1, U_2 \dots$ is a seq of measurable sets, $\bigcup_n U_n$ is measurable, $\bigcap_n U_n$ measurable

- Axioms for Lebesgue measure (read Tao)

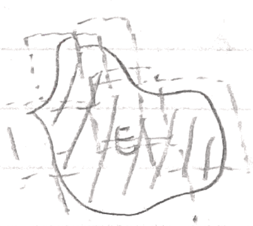
Thm: $\exists$ a definition of M_n and Lebesgue $m_n : M_n \to \mathbb{R}_{\geq 0}$ satisfying all axioms

- Defn: $\forall E \subset \mathbb{R}^n$

$$m^*(E) = \inf \Big\{ \sum_{n=1}^{\infty} |B_n| \,,\ \bigcup_{n=1}^{\infty} B_n \supset E,\ \underbrace{B_n \subset \mathbb{R}}_{\text{open boxes}} \Big\}$$

(outer measure)

Open box in $\mathbb{R}^1$: (a, b)
$\mathbb{R}^2$: $(a_1, b_1) \times (a_2, b_2)$
etc



$\Leftarrow$ cover E w/ countably open boxes

$\text{Vol}(B) = \prod (b_j - a_j)$

- outer measure well-defined $\forall$ subsets in $\mathbb{R}$

In discussion: prove properties of $m^*(E)$

Lemma 7.2.5; 7.2.6 : $m^*(\text{box}) = \text{vol}(\text{box})$

Lemma 7.2.5

(v) $m^*(\emptyset) = 0$

- 0 is less than the volume of any box
- for any $\varepsilon > 0$, box w/ dimensions $< \varepsilon^{\frac{1}{n}}$ covers $\emptyset$ and has vol less than ε

(vi) $0 \le m^*(\Omega) \le +\infty$

- box has pos volume

(vii) $A \subseteq B \subseteq \mathbb{R}^n$ ⇒ $m^*(A) \le m^*(B)$

- boxes covering B also cover A

(viii) If $(A_j)_{j \in J}$ finite collection of subsets of $\mathbb{R}^n$, $m^*(\bigcup_{j \in J} A_j) \le \sum_{j \in J} m^*(A_j)$

- union of boxes that cover each of the A_j must also cover union
- get rid of ε error by splitting it up amongst A_j

(x) countable sub-additivity - split epsilon by $\frac{\varepsilon}{2}, \frac{\varepsilon}{4}, \frac{\varepsilon}{8} \dots$ so they sum to finite