Math 105 HW 8

(8.3) Let $B = \prod_{i=1}^{n} [a_i, b_i]$. Suppose $(f_n) \to f$ a.e. on B. Then, $\forall \varepsilon > 0$, $\exists X$ s.t. $m(X^c) \le \varepsilon$ and $(f_n) \to f$ uniformly on X.

Proof:

Let

$$X(k, l) = \{x \in B : \forall n \ge k \text{ we have } |f_n(x) - f(x)| < 1/l\}$$

For a fixed $l \in \mathbb{N}$, since $f_n(x) \to f(x)$ a.e., $\bigcup_k X(k, l) \cup Z(l) = [a, b]$ for a zero set $Z(l)$.

Let $\varepsilon > 0$. Measure continuity implies $m(X(k, l)) \to b-a$ as $k \to \infty$ because $X(k, l) \to [a, b]$. So, we can pick $k_1 < k_2 < \dots$ s.t. for $X_l = X(k_l, l)$ $m(X_l^c) < \varepsilon/2^l \Rightarrow m(X^c) < \varepsilon$ where $X = \bigcap_l X_l$, as $X^c = \bigcup_l X_l^c$.

Claim: $f_n \to f$ uniformly on X. Let $\delta > 0$, $1/l < \delta$. $\forall n \ge k_l$,

$$x \in X \Rightarrow x \in X_l = X(k_l, l) \Rightarrow |f_n(x) - f(x)| < 1/l < \delta$$

So $f_n \to f$ uniformly on X^c.

b) True. We can follow the same proof technique, since each $X(k,l)$ will have finite measure as it is a subset of the domain, which has finite measure.

c) We can use the "moving bump" example, where $f_n(x) = 1$ if $x \in (n, n+1)$ and 0 otherwise. The sequence converges pointwise to 0, but on $\mathbb{R} \setminus B$ for any finite subinterval $B \subseteq \mathbb{R}$, does not converge uniformly to 0.

d) Let $\varepsilon > 0$. Let $B_i = [-i, i]^n$ for $i \in \mathbb{N}$. By Egoroff's theorem, for B_i $\exists S_i$ s.t. $m(S_i) < \frac{\varepsilon}{2^i}$ and $f_n \to f$ uniformly on $B_i \cap S_i^c$. Let $S = \bigcup_{i=1}^{\infty} S_i$. Note that $m(S) \leq \varepsilon$.

Now suppose K is compact. Then $\exists B_n$ s.t. $K \subseteq B_n$. We know that $f_n \to f$ uniformly on $B_n \cap S_n^c \supseteq B_n \cap S^c \supseteq K \cap S^c$. Thus $f_n \to f$ uniformly on $K \cap S^c$, showing the desired result.

(3) $\|T\|$ is the maximum singular value of T. To see why, we can write $T = U \Sigma V^T$, the SVD of T. Then $|\Sigma v|$ is maximized when $v = e_1$ (assuming Σ is arranged s.t. $\Sigma_{1,1}$ has the largest singular value). Since U, V^+ are unitary, this implies that the largest singular value is also the maximum of $|Tv|$ when $|v| \leq 1$.

(4) Holder's inequality:

The proof of Holder's inequality is an application of Young's inequality, which states that for $p, q \in (1, \infty)$, and $\frac{1}{p} + \frac{1}{q} = 1$, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$. We apply this to $a_j = \frac{x_j}{(\sum |x_j|^p)^{1/p}}$, $b_j = \frac{y_j}{(\sum |y_j|^q)^{1/q}}$.

Minkowski's inequality:

This is an application of the triangle inequality and Holder's inequality.