Math 105 HW 7

(39) First, note that $fg$ is measurable by Exercise 28.

Then, note that the set of measurable functions over $\mathbb{R}$ forms a vector space, as the constant function $0$ is measurable, the sum of measurable functions is measurable, scaling by a constant does not affect measurability, and the remaining axioms follow from usual properties of functions.

Then, the inner product defined by $\langle f, g \rangle = \int fg$ is a valid inner product because $\int fg = \int gf$, $\int (af+bg)h = a\int fh + b\int gh$ (Thm 34), and $\int f^2 \geq 0$.

Thus the result follows by Cauchy-Schwarz.

(48) For $x = \sum_{i=1}^{\infty} \frac{w_i}{3^i}$ we have that $\hat{H}(3^k x) = \hat{H}\left(\sum_{i=1}^{\infty} \frac{w_i}{3^{i-k}}\right) = \sum_{j=1}^{k} 3^{k-j} w_j + H\left(\sum_{i=k+1}^{\infty} \frac{w_i}{3^{i-k}}\right) \le 3^k$, so $J(x) \le \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k$ and $J(x)$ is finite.

Thus, by the Weierstrass M-test, $J(x)$ converges uniformly (and absolutely). Thus since each $\hat{H}(3^k x)$ is continuous, $J$ is continuous. $J$ is strictly increasing because each $H_k$ is strictly increasing.

(53) a) We have for the iterated integrals

$$\int\left[\int f_x(y)\,dy\right]dx = \int_0^1\left[\int_x^1 \frac{1}{y^2}\,dy + \int_0^x -\frac{1}{x^2}\,dy\right]dx$$

$$= \int_0^1 \left[-\frac{1}{y}\right]_x^1 + \left(-\frac{1}{x}\right) \quad dx$$

$$= \int_0^1 -1 + \frac{1}{x} - \frac{1}{x}\,dx$$

$$= \boxed{-1}$$

$$\int\left[\int f_y(x)\,dx\right]dy = \int_0^1\left[\int_0^y \frac{1}{y^2}\,dx + \int_y^1 -\frac{1}{x^2}\,dx\right]dy$$

$$= \int_0^1 \frac{1}{y} + \left[\frac{1}{x}\right]_y^1 \quad dy$$

$$= \int_0^1 \frac{1}{y} + 1 - \frac{1}{y}\,dy$$

$$= \boxed{1}$$

Since $f$ is non-negative, it is integrable if $f_\pm$ are integrable. We have

$$f_+ = \begin{cases} \frac{1}{y^2} & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_- = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

However, $f_\pm$ are not integrable. We show this for $f_+$ by constructing a set of disjoint boxes contained in $m^*(\mathcal{U}f)$ that have volume approaching infinity. For $n \in \mathbb{N}$, define $B_1, \dots B_n$ by

$$B_i = \left(0, \frac{i-1}{n}\right) \times \left(\frac{i-1}{n}, \frac{i}{n}\right) \times \left(0, \left(\frac{n}{i}\right)^2\right)$$

Note that the $B_i$ are disjoint because the $\left(\frac{i-1}{n}, \frac{i}{n}\right)$ are disjoint intervals. They are contained in $\mathcal{U}f$ because for each $(x, y, z) \in B_i$, $0 < x < y < 1$, and as $y < \frac{i}{n}$, $f(x,y) = \frac{1}{y^2} > \left(\frac{n}{i}\right)^2$.

Furthermore, $\operatorname{Vol}(B_i) = \frac{i-1}{n} \cdot \frac{1}{n} \cdot \left(\frac{n}{i}\right)^2 = \frac{i-1}{i^2} = \frac{1}{i} - \frac{1}{i^2}$

So $\sum_{i=1}^{n} \operatorname{Vol}(B_i) = \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i^2}$. As $n \to \infty$, this sum goes to $\infty$, so $m^*(\mathcal{U}f) = \infty$ and $f$ is not integrable. There fore the double integral does not exist.

b) Corollary 43 assumes that $f$ is non-negative, which is not the case here.

(58) a) The Lebesgue Density Theorem says that almost every point of $E$ is a density point. Every cube containing $x$ (not equal to $x$) contains a ball containing $x$, so a.e. point of $\bar{E}$ is also a balanced density point.

b) For each $(1/_{n+1}, 1/n)$ and $(-1/n, -1/_{n+1})$ pick a subinterval that is $\alpha(\frac{1}{n} - \frac{1}{n+1})$ long and take the union, and let $P=0$ (Not sure about this, just an idea)

66