Math 105 HW 6

8.3.2

a) If f is absolutely integrable, $\int_\Omega |f|$ is finite. By prop. 8.1.10, $\int_\Omega |cf| = \int_\Omega |c||f| = |c|\int_\Omega |f| < \infty$, so cf is absolutely integrable as desired.

Furthermore, we have that

$$\int_\Omega cf = \int_\Omega (cf)^+ - \int_\Omega (cf)^-$$

$$= \int_\Omega \max(cf, 0) - \int_\Omega -\min(cf, 0).$$

Note that

$$\max(cf, 0) = \begin{cases} c\max(f,0) & c \geq 0 \\ c\min(f,0) & c < 0 \end{cases}$$

$$\min(cf, 0) = \begin{cases} c\min(f,0) & c \geq 0 \\ c\max(f,0) & c < 0 \end{cases}$$

Thus, if $c \geq 0$, by prop. 8.1.10

$$\int_\Omega cf = \int_\Omega c\max(f,0) - \int_\Omega -c\min(f,0)$$

$$= c\int_\Omega \max(f,0) - c\int_\Omega -\min(f,0)$$

$$= c\int_\Omega f$$

Similarly, if $c < 0$,

$$\int_\Omega cf = \int_\Omega c\min(f,0) - \int -c\max(f,0)$$

$$= -c\int -\min(f,0) + c\int \max(f,0)$$

$$= c\int_\Omega f$$

as desired.

b) By the triangle inequality, $|f+g| \leq |f| + |g|$.
Thus, $\int_\Omega |f+g| \leq \int_\Omega |f| + |g| = \int_\Omega |f| + \int_\Omega |g|$
by Prop 8.2.6 and Lemma 8.2.10, so
$f+g$ is absolutely integrable.

Then,

$$\int_\Omega f+g = \int_\Omega \max(f+g,0) - \int_\Omega -\min(f+g,0)$$

$$\int_\Omega f = \int_\Omega \max(f,0) - \int_\Omega -\min(f,0)$$

$$\int_\Omega g = \int_\Omega \max(g,0) - \int_\Omega -\min(g,0)$$

Note that

$$\max(f+g,0) = \begin{cases} \max(f,0) + \max(g,0), & f,g \geq 0 \text{ or } f,g \leq 0 \\ \max(f,0) + \min(g,0), & f, f+g \geq 0 \text{ \& } g \leq 0 \text{ or } f, f+g \leq 0 \text{ \& } g \geq 0 \\ \max(g,0) + \min(f,0), & g, f+g \geq 0 \text{ \& } f \leq 0 \text{ or } g, f+g \leq 0 \text{ \& } f \geq 0 \end{cases}$$

$$\min(f+g,0) = \begin{cases} \min(f,0) + \min(g,0), & f,g \geq 0 \text{ or } f,g \leq 0 \\ \min(f,0) + \max(g,0), & f, f+g \geq 0 \text{ \& } g \leq 0 \text{ or } f, f+g \leq 0 \text{ \& } g \geq 0 \\ \min(g,0) + \max(f,0), & g, f+g \geq 0 \text{ \& } f \leq 0 \text{ or } g, f+g \leq 0 \text{ \& } f \geq 0 \end{cases}$$

Thus, if $f, g \geq 0$ or $f, g \leq 0$

$$\begin{aligned}\int f+g &= \int \max(f,0) + \max(g,0) \\ &\quad - \int -\min(f,0) - \min(g,0) \\ &= \int \max(f,0) - \int -\min(f,0) \\ &\quad + \int \max(g,0) - \int -\min(g,0) \\ &= \int f + \int g\end{aligned}$$

If $f, f+g \geq 0$ + $g \leq 0$ or $f, f+g \leq 0$ + $g \geq 0$

$$\begin{aligned}\int f+g &= \int \max(f,0) - (-\min(g,0)) \\ &\quad - \int -\max(g,0) + (-\min(f,0)) \\ &= \int \max(f,0) - \int -\min(f,0) \\ &\quad + \int \max(g,0) - \int -\min(g,0) \\ &= \int f + \int g\end{aligned}$$

If $g, f+g \geq 0$ + $f \leq 0$ or $g, f+g \leq 0$ + $f \geq 0$

$$\begin{aligned}\int f+g &= \int \max(g,0) - (-\min(f,0)) \\ &\quad - \int -\max(f,0) + (-\min(g,0)) \\ &= \int \max(f,0) - \int -\min(f,0) \\ &\quad + \int \max(g,0) - \int -\min(g,0) \\ &= \int f + \int g\end{aligned}$$

Where the simplifications follow from a) and Lemma 8.2.10. This shows the desired as we have covered all possible cases.

c) We have

$$\int_\Omega f = \int_\Omega \max(f,0) - \int_\Omega -\min(f,0)$$
$$\int_\Omega g = \int_\Omega \max(g,0) - \int_\Omega -\min(g,0)$$

Note that since $f \geq g$, $\max(f,0) \geq \max(g,0)$, and $\min(f,0) \geq \min(g,0) \Leftrightarrow -\min(f,0) \leq -\min(g,0)$.

Thus, by Prop. 8.2.6, $\int_\Omega \max(f,0) \geq \int_\Omega \max(g,0)$ and $\int_\Omega -\min(f,0) \leq \int_\Omega -\min(g,0)$. Combining the inequalities shows the result.

d) If $f(x) = g(x)$ a.e., then $\max(f,0) = \max(g,0)$ and $-\min(f,0) = -\min(g,0)$ a.e. Then, by Prop 8.2.6, $\int \max(f,0) = \int \max(g,0)$ and $\int -\min(f,0) = \int -\min(g,0)$. Subtracting the two equalities shows that $\int_\Omega f = \int_\Omega g$.

8.3.3

Note that $0 \leq g(x) - f(x) \quad \forall x \in \mathbb{R}$,
and by Prop 8.3.3 a), b),
$0 = \int_{\mathbb{R}} g - \int_{\mathbb{R}} f = \int_{\mathbb{R}} g - f$. Then, since
$g - f$ is non-negative, Prop. 8.2.6 a)
implies that $g(x) - f(x) = 0 \iff g(x) = f(x)$
for almost every $x \in \mathbb{R}$, as desired.