# Math 105 HW 2

## Lemma 0

By countable sub-additivity,
$m^*(E \cup F) \leq m^*(E) + m^*(F)$

Let $\varepsilon > 0$. Let $(B_j)_{j \in J}$ be a covering of $E \cup F$ s.t. $\sum_{j \in J} \mathrm{vol}(B_j) \leq m^*(E \cup F) + \varepsilon$.
Let $\delta = \mathrm{dist}(E, F)$ and let
$A_E = \{ x \in \mathbb{R}^d \mid \inf\{|x-y|, y \in E\} < \frac{\delta}{2} \}$
$A_F = \{ x \in \mathbb{R}^d \mid \inf\{|x-y|, y \in F\} < \frac{\delta}{2} \}$

Note that $A_E \cap A_F = \emptyset$ and $A_E$, $A_F$ are both open, thus so are $A_E \cap (B_j)_{j \in J}$ and $A_F \cap (B_j)_{j \in J}$, so we may write them as countable unions of open boxes $(B_i)_{i \in I}$, $(B_k)_{k \in K}$. Thus, since $E \subseteq A_E \cap (B_j)_{j \in J}$, $F \subseteq A_F \cap (B_j)_{j \in J}$, we have

$$m^*(E) + m^*(F) \leq \sum_{i \in I} \mathrm{Vol}(B_i) + \sum_{k \in K} \mathrm{Vol}(B_k)$$

$$\leq \sum_{j \in J} \mathrm{Vol}(B_j)$$

$$\leq m^*(E \cup F) + \varepsilon$$

This shows the reverse inequality, so
$m^*(E \cup F) = m^*(E) + m^*(F)$

Lemma 1

Let $\varepsilon > 0$. Pick $(B_j)_{j \in J}$ covering $A$ s.t. $\sum_{j \in J} vol(B_j) < m^*(A) + \varepsilon$. Since $(B_j)_{j \in J}$ is open,

$$\inf\{m^*(U) \mid U \supset A, U \text{ open}\} \leq$$
$$m^*((B_j)_{j \in J}) \leq$$
$$\sum_{j \in J} vol(B_j) <$$
$$m^*(A) + \varepsilon$$

Thus, $\inf\{m^*(U) \mid U \supset A, U \text{ open}\} \leq m^*(A)$

Now pick an open set $U$ s.t.
$m^*(U) < \inf\{m^*(U) \mid U \supset A, U \text{ open}\} + \varepsilon$
and $A \subset U$. By monotonicity,
$$m^*(A) \leq m^*(U) < \inf\{m^*(U) \mid U \supset A, U \text{ open}\} + \varepsilon$$

This shows the reverse inequality, so
$\inf\{m^*(U) \mid U \supset A, U \text{ open}\} = m^*(A)$.

Lemma 2

Let $\varepsilon > 0$. For each $i$, we may find an open set $\mathcal{U}_i$ s.t. $\mathcal{U}_i \supset E_i$ and $m^*(\mathcal{U}_i \setminus E_i) < \frac{\varepsilon}{2^i}$. Note that $\bigcup_i \mathcal{U}_i \supset \bigcup_i E_i$ and $\bigcup_i \mathcal{U}_i$ is open. Furthermore, $\bigcup_i \mathcal{U}_i \setminus \bigcup_i E_i \subseteq \bigcup_i (\mathcal{U}_i \setminus E_i)$, so by monotonicity and countable subadditivity,

$$\begin{aligned} m^*(\textstyle\bigcup_i \mathcal{U}_i \setminus \bigcup_i E_i) &\leq m^*(\textstyle\bigcup_i (\mathcal{U}_i \setminus E_i)) \\ &\leq \textstyle\sum_i m^*(\mathcal{U}_i \setminus E_i) \\ &< \textstyle\sum_i \frac{\varepsilon}{2^i} = \varepsilon \end{aligned}$$

which shows that $\bigcup_i E_i$ is measurable.

Lemma 3

Let $A \subseteq \mathbb{R}^n$ be closed. For $i \in \mathbb{N}$, let $V_i = A \cap [-i, i]^n$. Since $A$ is closed and $[-i, i]^n$ is closed and bounded, $V_i$ is closed and bounded. Furthermore, $A = \bigcup_{i=1}^{\infty} V_i$, so $A$ can be written as a countable union of closed + bounded sets.

We now claim any bounded closed set is measurable. Let $A \subseteq \mathbb{R}^n$ be closed, $\varepsilon > 0$. and bounded. By Lemma 1, we may find an open set $\mathcal{U}$ s.t. $A \subseteq \mathcal{U}$ and $m^*(\mathcal{U}) < m^*(A) + \varepsilon$. Note that $\mathcal{U} \setminus A$ is open.

First we show that for $K \subseteq \mathcal{U} \setminus A$ closed, $m^*(K) + m^*(A) = m^*(A \cup K) \leq m^*(\mathcal{U})$. AFSOC $\operatorname{dist}(K, A) = 0$. Then, $\forall n \in \mathbb{N}$, we may find $x_n \in K$ s.t. $\inf \{ |x_n - y| : y \in A \} < \frac{1}{n}$. Since $K$ is compact, $\{x_n\}$ has a convergent subsequence converging to some $x \in K$. Furthermore, $\inf \{ |x - y| : y \in A \} = 0$. Similarly, we can construct a sequence $\{y_n\} \subseteq A$ by choosing $y_n \in A$ for $n \in \mathbb{N}$ s.t. $|x - y_n| < \frac{1}{n}$. Then $\lim_{n \to \infty} y_n = x \in A$ as $A$ is compact. But then $x \in A$ and $x \in \mathcal{U} \setminus A$, a contradiction. So $\delta = \operatorname{dist}(K, A) > 0$.

Now, we construct an increasing sequence of closed subsets $K_m \subseteq \mathcal{U}\setminus A$ as follows: Divide $\mathbb{R}^n$ into the boxes $\prod_{i=0}^{m} [\frac{q_i}{2^i}, \frac{q_i+1}{2^i}]^n$ for $q \in \mathbb{Z}$. Let $K = \bigcup B^{(1)}$ be the union of all the unit boxes contained in $\mathcal{U}\setminus A$. For $m > 1$, let $K_m = K_{m-1} \cup (\bigcup B^{(m)})$, where $(B^{(m)})$ is the union of all boxes of dimension $\frac{1}{2^{m-1}}$ contained in $(\mathcal{U}\setminus A)\setminus K_{m-1}$.

Since each $K_m$ is a union of finitely many compact sets, it is compact, so by the previous $m^*(K_m) + m^*(A) \le m^*(\mathcal{U})$ $\Leftrightarrow m^*(K_m) \le m^*(\mathcal{U}) - m^*(A) < \varepsilon$. Thus, $\lim_{m\to\infty} m^*(K_m) < \varepsilon$.

Furthermore, $\lim_{m\to\infty} K_m = \mathcal{U}\setminus A$ because since $\mathcal{U}\setminus A$ is open, for $x \in \mathcal{U}\setminus A$ we can find some box containing $x$ within $\mathcal{U}\setminus A$. We claim $m^*(K_m) = \sum_{j\in J} m^*(B_j)$ where $J$ is the set of boxes that form the union of boxes that is $K_m$. For $\varepsilon>0$, we can shrink $B_j$ on all sides to get $B_j'$ where $\mathrm{vol}(B_j') = \mathrm{vol}(B_j) + \frac{\varepsilon}{2^j}$. Then the $B_j'$ are disjoint w/ positive distance so by Lemma 0 $m^*(\bigcup_{j\in J} B_j') = \sum_{j\in J} m^*(B_j') = \sum_{j\in J} m^*(B_j) - \varepsilon \le m^*(K_m)$. Thus $\sum_{j\in J} m^*(B_j) \le m^*(K_m)$ and the reverse follows from subadditivity.

Thus $m^*(\mathcal{U}\setminus A) = m^*(\lim_{m\to\infty} K_m) = m^*(\bigcup_{m=1}^{\infty}(B^{(m)})) \le \sum_{m=1}^{\infty} m^*(B^{(m)})$ $= \lim_{n\to\infty} m^*(K_m) < \varepsilon$ as desired.

Lemma 4

Since $E$ is measurable, for $i \in \mathbb{N}$, we can find and open set $\mathcal{U}_i \supset E$ s.t. $m^*(\mathcal{U}_i \setminus E) < 1/i$. Note that for each $i$, $\mathbb{R}^n \setminus \mathcal{U}_i$ is closed, so $V = \bigcup_{i=1}^{\infty} (\mathbb{R}^n \setminus \mathcal{U}_i) \subseteq E^c$ is a countable union of closed sets.

We claim that $m^*(E^c \setminus V) = 0$. Let $\varepsilon > 0$. Pick $m$ s.t. $\frac{1}{m} < \varepsilon$. Then, note that $E^c \setminus V \subseteq E^c \setminus (\mathbb{R}^n \setminus \mathcal{U}_m) = \mathcal{U}_m \setminus E$. By monotonicity, $m^*(E^c \setminus V) \leq m^*(\mathcal{U}_m \setminus E) < 1/m < \varepsilon$. Thus $m^*(E^c \setminus V) = 0$ since $\varepsilon$ was arbitrary.

Finally, note that $E^c = V \cup E^c \setminus V$. Since $V$ is a countable union of closed sets, by lemmas 2 and 3 $V$ is measurable. Since $E^c \setminus V$ has outer measure $0$, it is measurable (for $\varepsilon > 0$, pick any open covering $B$ of $E^c \setminus V$ w/ vol $< \varepsilon$, and monotonicity implies $m^*(B \setminus (E^c \setminus V)) \leq m^*(B) < \varepsilon$). Thus by lemma 2 $E^c$ is measurable.