Math 105 HW 12

Let $w = f\,dx \wedge dy$ + $g\,dx \wedge dz$ + $h\,dy \wedge dz$ be a closed differential 2-form in $\mathbb{R}^3$.

Define cone map $p: \mathbb{R}^4 \to \mathbb{R}^3$ by $p(x, y, z, t) = (tx, ty, tz)$

Define $N: \Omega^2(\mathbb{R}^4) \to \Omega(\mathbb{R}^3)$ to map a 2-form β to $\int_0^1 \beta'$ where β' only has the terms with dt.

Claim: $d\,p^*(w) = p^*(dw) = 0$.

Note that since w is closed, $dw = 0$ $\Rightarrow p^*(dw) = 0$.

Furthermore,

$$p^*(w) = w \circ p^*$$
$$= f(tx, ty, tz)\, dtx \wedge dty + g(tx, ty, tz)\, dtx \wedge dtz + h(tx, ty, tz)\, dty \wedge dtz$$
$$= f(tx, ty, tz)(t^2 dx \wedge dy + t\,dt \wedge (x\,dy - y\,dx)) + g(tx, ty, tz)(t^2 dx \wedge dz + t\,dt \wedge (x\,dz - z\,dx)) + h(tx, ty, tz)(t^2 dy \wedge dz + t\,dt \wedge (y\,dz - z\,dy))$$

So by commutativity of pullback,

$d p^*(w) = 0 \Rightarrow$

$$d[f_t\, t\,dt \wedge (x\,dy - y\,dx) + g_t\, t\,dt(x\,dz - z\,dx) + h_t\, t\,dt \wedge (y\,dz - z\,dy)] = - d[f_t t^2 dx \wedge dy + g_t t^2 dx \wedge dz + h_t t^2 dy \wedge dz]$$

(where $f_t = f(tx, ty, tz)$ and similar for g_t, h_t).

Then,

$N \circ p^*(\omega) =$

$\int_0^1 f_t\, t\, dt \wedge (x\,dy - y\,dx) + \int_0^1 g_t\, t\, dt \wedge (x\,dz - z\,dx)$

$+ \int_0^1 h_t\, t\, dt \wedge (y\,dz - z\,dy)$

$d(N \circ p^*(\omega)) =$
$d[\int_0^1 f_t\, t\, dt \wedge (x\,dy - y\,dx)] + d[\int_0^1 g_t\, t\, dt \wedge (x\,dz - z\,dx)]$
$+ d[\int_0^1 h_t\, t\, dt \wedge (y\,dz - z\,dy)] =$

$[d\int_0^1 f_t\, t\, dt \wedge (x\,dy - y\,dx) + d\int_0^1 g_t\, t\, dt \wedge (x\,dz - z\,dx)$
$+ d\int_0^1 h_t\, t\, dt \wedge (y\,dz - z\,dy)] -$
$[\int_0^1 f_t\, t\, dt \wedge d(x\,dy - y\,dx) + \int_0^1 g_t\, t\, dt \wedge d(x\,dz - z\,dx)$
$+ \int_0^1 h_t\, t\, dt \wedge d(y\,dz - z\,dy)] =$

$\int_0^1 df_t\, t\, dt \wedge (x\,dy - y\,dx) + dg_t\, t\, dt \wedge (x\,dz - z\,dx)$
$+ dh_t\, t\, dt \wedge (y\,dz - z\,dy) -$
$[\int_0^1 f_t\, t\, dt \wedge d(x\,dy - y\,dx) + \int_0^1 g_t\, t\, dt \wedge d(x\,dz - z\,dx)$
$+ \int_0^1 h_t\, t\, dt \wedge d(y\,dz - z\,dy)] =$

$-\int_0^1 [df_t\, t^2\, dx \wedge dy + dg_t\, t^2\, dx \wedge dz + dh_t\, t^2\, dy \wedge dz]$
$+ [\int_0^1 f_t\, t\, dt \wedge d(x\,dy - y\,dx) + \int_0^1 g_t\, t\, dt \wedge d(x\,dz - z\,dx)$
$+ \int_0^1 h_t\, t\, dt \wedge d(y\,dz - z\,dy)]$

Then since $d(x\,dy - y\,dx) = d(x\,dz - z\,dx)$
$= d(y\,dz - z\,dy) = 0$

$$d(N \circ p^*(\omega)) = -\int_0^1 [d f_t\, t^2\, dx \wedge dy + d g_t\, t^2\, dx \wedge dz + d h_t\, t^2\, dy \wedge dz]$$

$$= -\Big[f_t\, t^2\, dx \wedge dy + g_t\, t^2\, dx \wedge dz + h_t\, t^2\, dy \wedge dz\Big]_0^1$$

$$= -(f\, dx \wedge dy + g\, dx \wedge dz + h\, dy \wedge dz)$$
$$= -\omega$$

Thus
$$(N \circ p^* d + d\, N \circ p^*)\,\omega = -\omega$$
$$\omega = -d\, N \circ p^*(\omega)$$
$$= d(-N \circ p^*(\omega))$$

So ω is exact.