Math 105 HW 11

We have, using the fact that $d(f\,dx + g\,dy) = (g_x - f_y)\,dx \wedge dy$,

$$d\Omega_1 = d(|x|^{-2}(x_1\,dx_2 - x_2\,dx_1))$$
$$= d(|x|^{-2}x_1\,dx_2 - |x|^{-2}x_2\,dx_1)$$
$$= d\left(\frac{x_1}{x_1^2+x_2^2}\,dx_2 - \frac{x_2}{x_1^2+x_2^2}\,dx_1\right)$$
$$= \left(\frac{2x_1x_2}{x_1^2+x_2^2} - \frac{2x_1x_2}{x_1^2+x_2^2}\right) dx_1 \wedge dx_2$$
$$= 0$$

Similarly,

$$d\Omega_2 = d(|x|^{-3}(x_1\,dx_2 \wedge dx_3 - x_2\,dx_1 \wedge dx_3 + x_3\,dx_1 \wedge dx_2))$$
$$= d(|x|^{-3}x_1\,dx_{23} - |x|^{-3}x_2\,dx_{13} + |x|^{-3}x_3\,dx_{12})$$
$$= d(|x|^{-3}x_1) \wedge dx_{23} - d(|x|^{-3}x_2) \wedge dx_{13} + d(|x|^{-3}x_3) \wedge dx_{23}$$

$d(|x|^{-3}x_1) \wedge dx_{23} =$
$[(|x|^{-3} - 3x_1^2|x|^{-5})\,dx_1 - 3x_1x_2|x|^{-5}\,dx_2 - 3x_1x_3|x|^{-5}\,dx_3)] \wedge dx_{23} =$
$(|x|^{-3} - 3x_1^2|x|^{-5})\,dx_1 \wedge dx_2 \wedge dx_3$

$d(|x|^{-3}x_2) \wedge dx_{13} =$
$[-3x_1x_2|x|^{-5}\,dx_1 + (|x|^{-3} - 3x_2^2|x|^{-5})\,dx_2 - 3x_2x_3|x|^{-5}\,dx_3] \wedge dx_{13} =$
$(|x|^{-3} - 3x_2^2|x|^{-5})\,dx_2 \wedge dx_1 \wedge dx_3$

$d(|x|^{-3}x_3) \wedge dx_{12} =$
$[-3x_1x_3|x|^{-5}\,dx_1 + 3x_2x_3|x|^{-5}\,dx_2 + (|x|^{-3} - 3x_3^2|x|^{-5})\,dx_3) \wedge dx_{12} =$
$(|x|^{-3} - 3x_3^2|x|^{-5})\,dx_3 \wedge dx_1 \wedge dx_2$

(using that $dx \wedge dx = 0$).

Thus by signed commutativity

$$\begin{aligned}
d\Omega_2 &= (|x|^{-3} - 3x_1^2|x|^{-5})\, dx_1 \wedge dx_2 \wedge dx_3 \\
&\quad - (|x|^{-3} - 3x_2^2|x|^{-5})\, dx_2 \wedge dx_1 \wedge dx_3 \\
&\quad + (|x|^{-3} - 3x_3^2|x|^{-5})\, dx_3 \wedge dx_1 \wedge dx_2 \\
&= (|x|^{-3} - 3x_1^2|x|^{-5})\, dx_1 \wedge dx_2 \wedge dx_3 \\
&\quad + (|x|^{-3} - 3x_2^2|x|^{-5})\, dx_1 \wedge dx_2 \wedge dx_3 \\
&\quad + (|x|^{-3} - 3x_3^2|x|^{-5})\, dx_1 \wedge dx_2 \wedge dx_3 \\
&= (3|x|^{-3} - 3(x_1^2 + x_2^2 + x_3^2)|x|^{-5})\, dx_1 \wedge dx_2 \wedge dx_3 \\
&= (3|x|^{-3} - 3|x|^2|x|^{-5})\, dx_1 \wedge dx_2 \wedge dx_3 \\
&= 0
\end{aligned}$$

For the general case, we can define

$$\Omega_n = |x|^{-n}\left(\sum_{i=1}^{n} (-1)^{i+1} x_i\, dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_n \right)$$

Then, when we compute $d\Omega_n$ consider each

$$d(|x|^{-n} x_i) \wedge dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_n$$

we have

$$\begin{aligned}
d(|x|^{-n} x_i) &= x_i\left(-\tfrac{n}{2}\right)|x|^{-n-2} \cdot 2x_1\, dx_1 + \dots \\
&\quad + \left(|x|^{-n} + x_i\left(-\tfrac{n}{2}\right)|x|^{-n-2} \cdot 2x_i\right) dx_i + \dots \\
&\quad + x_i\left(-\tfrac{n}{2}\right)|x|^{-n-2} \cdot 2x_n\, dx_n \\
&= -n\, x_1 x_i |x|^{-n-2}\, dx_1 + \dots + \\
&\quad (|x|^{-n} - n x_i^2 |x|^{-n-2})\, dx_i + \dots \\
&\quad - n x_n x_i |x|^{-n-2}\, dx_n
\end{aligned}$$

Thus

$$d(|x|^{-n} x_i) \, dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_n =$$
$$(|x|^{-n} - n x_i^2 |x|^{-n-2}) \wedge dx_i \wedge dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_n$$
$$= (-1)^{i+1} (|x|^{-n} - n x_i^2 |x|^{-n-2}) d_1 \wedge \dots \wedge d_n$$

$$d\Omega_n = \sum_{i=1}^{n} d(|x|^{-n} x_i) \, dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_n =$$

$$\sum_{i=1}^{n} |x|^{-n} - n x_i^2 |x|^{-n-2} =$$

$$n|x|^{-n} - n|x|^{-n-2}(x_1^2 + \dots + x_n^2) =$$
$$n|x|^{-n} - n|x|^{-n-2}|x|^2 =$$
$$0$$

as desired.

Parametrized unit sphere

$$\int_\gamma \Omega_2 = \int_\gamma |x|^{-3}\left(x_1\,dx_2\wedge dx_3 - x_2\,dx_1\wedge dx_3 + x_3\,dx_1\wedge dx_2\right).$$

$$= \int_\gamma |x|^{-3}x_1\,dx_2\wedge dx_3 - \int_\gamma |x|^{-3}x_2\,dx_1\wedge dx_3 + \int_\gamma |x|^{-3}x_3\,dx_1\wedge dx_2$$

$$\frac{dx_1}{ds} = \cos(2\pi t)\cdot\pi\cos(\pi s) \qquad \frac{dx_1}{dt} = -2\pi\sin(2\pi t)\cdot\sin(\pi s)$$

$$\frac{dx_2}{ds} = \sin(2\pi t)\cdot\pi\cos(\pi s) \qquad \frac{dx_2}{dt} = 2\pi\cos(2\pi t)\cdot\sin(\pi s)$$

$$\frac{dx_3}{ds} = -\pi\sin(\pi s) \qquad \frac{dx_3}{dt} = 0$$

$$\int_\gamma |x|^{-3}x_1\,dx_2\wedge dx_3 = \int_0^1\int_0^1 \sin(\pi s)\cos(2\pi t)\begin{vmatrix}\frac{dx_2}{ds} & \frac{dx_2}{dt}\\ \frac{dx_3}{ds} & \frac{dx_3}{dt}\end{vmatrix}\,ds\,dt$$

$$= \int_0^1\int_0^1 \sin(\pi s)\cos(2\pi t)\left[\sin(2\pi t)\cdot\pi\cos(\pi s)\cdot 0 - 2\pi\cos(2\pi t)\cdot\sin(\pi s)\cdot(-\pi\sin(\pi s))\right]ds\,dt$$

$$= \int_0^1\int_0^1 2\pi^2\sin(\pi s)^3\cos(2\pi t)^2\,ds\,dt$$

$$\int_\gamma |x|^{-3}x_2\,dx_1\wedge dx_3 = \int_0^1\int_0^1 \sin(\pi s)\sin(2\pi t)\begin{vmatrix}\frac{dx_1}{ds} & \frac{dx_1}{dt}\\ \frac{dx_3}{ds} & \frac{dx_3}{dt}\end{vmatrix}\,ds\,dt$$

$$= \int_0^1\int_0^1 \sin(\pi s)\sin(2\pi t)\left[\cos(2\pi t)\cdot\pi\cos(\pi s)\cdot 0 - (-2\pi\sin(2\pi t)\cdot\sin(\pi s)\cdot(-\pi\sin\pi s))\right]ds\,dt$$

$$= \int_0^1\int_0^1 -2\pi^2\sin(\pi s)^3\sin(2\pi t)^2\,ds\,dt$$

$$\int_\gamma |x|^{-3} x_3 \, dx_1 \wedge dx_2 = \int_0^1 \int_0^1 \cos(\pi s) \begin{vmatrix} \frac{dx_1}{ds} & \frac{dx_1}{dt} \\ \frac{dx_2}{ds} & \frac{dx_2}{dt} \end{vmatrix} ds\, dt$$

$$= \int_0^1 \int_0^1 \cos(\pi s) \Big[\cos(2\pi t) \cdot \pi \cos(\pi s) \cdot 2\pi \cos(2\pi t) \cdot \sin(\pi s))$$
$$- (-2\pi \sin(2\pi t) \cdot \sin(\pi s) \cdot \sin(2\pi t) \cdot \pi \cos(\pi s))\Big]$$
$$ds\, dt$$

$$= \int_0^1 \int_0^1 \cos(\pi s)\Big[2\pi^2 \cos(2\pi t)^2 \cos(\pi s) \sin(\pi s) +$$
$$2\pi^2 \sin(2\pi t)^2 \cos(\pi s) \sin(\pi s))\Big] \, ds\, dt$$

$$= \int_0^1 \int_0^1 2\pi^2 \cos(\pi s)^2 \sin(\pi s) \, ds\, dt$$

$\leadsto$

$$\int_\gamma \Omega_2 = \int_0^1 \int_0^1 2\pi^2 \sin(\pi s)^3 \cos(2\pi t)^2 + 2\pi^2 \sin(\pi s)^3 \sin(2\pi t)^2$$
$$+ 2\pi^2 \cos(\pi s)^2 \sin(\pi s) \, ds\, dt$$

$$= 2\pi^2 \int_0^1 \int_0^1 \sin(\pi s)^3 + \cos(\pi s)^2 \sin(\pi s) \, ds\, dt$$

$$= 2\pi^2 \int_0^1 \int_0^1 \sin(\pi s) \, ds\, dt$$

$$= 2\pi^2 \int_0^1 \left[- \frac{\cos(\pi s)}{\pi} \right]_0^1 dt$$

$$= 2\pi^2 \int_0^1 \frac{2}{\pi} \, dt$$

$$= 2\pi^2 \cdot \frac{2}{\pi}$$

$$= \boxed{4\pi}$$

Stereographic projection

$$\frac{dx_1}{da} = \frac{2}{1+a^2+b^2} - \frac{4a^2}{(1+a^2+b^2)^2} \qquad \frac{dx_1}{db} = -\frac{4ab}{(1+a^2+b^2)^2}$$

$$\frac{dx_2}{da} = \frac{-4ab}{(1+a^2+b^2)^2} \qquad \frac{dx_2}{db} = \frac{2}{1+a^2+b^2} - \frac{4b^2}{(1+a^2+b^2)^2}$$

$$\frac{dx_3}{da} = \frac{2a}{1+a^2+b^2} - \frac{2a(-1+a^2+b^2)}{(1+a^2+b^2)^2} \qquad \frac{dx_3}{db} = \frac{2b}{1+a^2+b^2} - \frac{2b(-1+a^2+b^2)}{(1+a^2+b^2)^2}$$

$$= \frac{4a}{(1+a^2+b^2)^2} \qquad = \frac{4b}{(1+a^2+b^2)^2}.$$

$$\int_\gamma |x|^{-3} x_1\, dx_2 \wedge dx_3 = \int_{\mathbb{R}^2} \frac{2a}{1+a^2+b^2}\left[\frac{-16ab^2}{(1+a^2+b^2)^4} - \frac{8a+8a^3-8ab^2}{(1+a^2+b^2)^2}\right] da\,db$$

$$-\int_{\mathbb{R}^2} 2a \cdot \frac{8a+8a^3+8ab^2}{(1+a^2+b^2)^5}\, da\,db$$

$$\int_\gamma |x|^{-3} x_2\, dx_1 \wedge dx_3 = \int_{\mathbb{R}^2} \frac{2b}{1+a^2+b^2}\left(\frac{8b-8a^2b+8b^3}{(1+a^2+b^2)^4} + \frac{16a^2b}{(1+a^2+b^2)^4}\right)$$

$$da\,db$$

$$= \int_{\mathbb{R}^2} 2b \cdot \frac{8b+8b^3+8a^2b}{(1+a^2+b^2)^5}\, da\,db$$

$$\int_\gamma |x|^{-3} x_3\, dx_1 \wedge dx_2 = \int_{\mathbb{R}^2} \frac{-1+a^2+b^2}{1+a^2+b^2}\left[\frac{(2-2a^2+2b^2)(2+2a^2-2b^2)}{(1+a^2+b^2)^4}\right.$$

$$\left. - \frac{16a^2b^2}{(1+a^2+b^2)^4}\right] da\,db.$$

$$= \int_{\mathbb{R}^2} -1+a^2+b^2 \cdot \left(\frac{-4a^4-4b^4-8a^2b^2+4}{(1+a^2+b^2)^5}\right) da\,db$$

(used calculator to factor)

$\leadsto \int_\gamma \Omega_2 = \int_{\mathbb{R}^2} \frac{-4(1+a^2+b^2)^3}{(1+a^2+b^2)^5}$

$= -4 \int_{\mathbb{R}^2} \frac{1}{(1+a^2+b^2)^2}$

$= -4 \int_0^{2\pi} \int_0^\infty \frac{r}{(1+r^2)^2} \, dr \, d\theta$ (inspired by Jianzhi)

$= -4 \int_0^{2\pi} \left[-\frac{1}{2} \cdot \frac{1}{1+r^2} \right]_0^\infty d\theta$

$= -4 \int_0^{2\pi} \frac{1}{2} \, d\theta$

$= -4 \cdot \frac{1}{2} \cdot 2\pi$

$= \boxed{-4\pi}$