Math 105 HW 1

7.2.1 (v) Note that since volume is non-negative, $m^*(\emptyset) \geq 0$. Furthermore, $\forall\ \varepsilon > 0$, the box $B = \prod_{i=1}^{n} (0, \frac{\varepsilon^{1/n}}{2})$ has volume less than ε and covers $\emptyset$, so $m^*(\emptyset) < \varepsilon$. This shows that $m^*(\emptyset) \leq 0$ so $m^*(\emptyset) = 0$.

(vi) Clearly $m^*(\Omega) \leq +\infty$. Since volume is non-negative, $m^*(\Omega) \geq 0$.

(vii) If $A \subseteq B$, any open covering of B also covers A, i.e.

$$\{\sum_{j \in J} \mathrm{vol}(B_j) : (B_j)_{j \in J} \text{ covers } B, J \text{ countable}\} \subseteq$$

$$\{\sum_{j \in J} \mathrm{vol}(B_j) : (B_j)_{j \in J} \text{ covers } A, J \text{ countable}\}$$

Thus, since $S_1 \subseteq S_2 \subseteq \mathbb{R} \Rightarrow \inf(S_1) \geq \inf(S_2)$, as a lower bound on S_2 is also a lower bound on S_1, $m^*(A) \leq m^*(B)$.

(viii) Suppose $|J| = n$. Let $\varepsilon > 0$. For $j \in J$ we may find a countable covering $(B_i)_{i \in I_j}$ of A_j $\sum_{i \in I_j} \mathrm{vol}(B_i) < m^*(A_j) + \frac{\varepsilon}{n}$. Then, since a countable union of countable sets is countable, $(B_i)_{i \in I}$, where $I = \bigcup_{j \in J} I_j$ is a countable covering of $\bigcup_{j \in J} A_j$, and $\sum_{i \in I} \mathrm{vol}(B_i) < \sum_{j \in J} m^*(A_j) + \varepsilon$. This shows that $m^*(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m^*(A_j)$ as desired. (Note that if $\sum_{j \in J} m^*(A_j) = +\infty$, the result follows from positivity.

(x) Let $\varepsilon > 0$. Since J is countable, we may number $J = \{j_1, j_2, \ldots\}$. For $j_m \in J$, we may find a countable covering $(B_i)_{i \in I_m}$ of A_{j_m} such that $\sum_{i \in I_m} \mathrm{vol}(B_i) < m^*(A_{j_m}) + \frac{\varepsilon}{2^m}$. Then, since a countable union of countable sets is countable, $(B_i)_{i \in I}$, where $I = \bigcup_{m \in \mathbb{N}} I_m$ is a countable covering of $\bigcup_{j \in J} A_j$. Furthermore,

$$\sum_{i \in I} \mathrm{vol}(B_i) < \sum_{j \in J} m^*(A_j) + \varepsilon \sum_{m \in \mathbb{N}} \frac{1}{2^m}$$
$$= \sum_{j \in J} m^*(A_j) + \varepsilon$$

so $m^*\left(\bigcup_{j \in J} A_j\right) \leq \sum_{j \in J} m^*(A_j)$ as desired.

(xiii) Suppose $(B_j)_{j \in J}$ is a countable covering of Ω. Then, $(x + B_j)_{j \in J}$ is a countable covering of $x + \Omega$, since for $y \in \Omega$, $y \in \prod_{i=1}^{n} (a_i, b_i)$, $y + x \in \prod_{i=1}^{n} (a_i + x_i, b_i + x_i)$. Similarly, for $(B_j)_{j \in J}$ a countable covering of $x + \Omega$, $(-x + B_j)_{j \in J}$ is a countable covering of Ω. Thus, there is a bijection between countable coverings of Ω and $x + \Omega$. Since $\sum_{j \in J} \mathrm{Vol}(B_j) = \sum_{j \in J} \mathrm{Vol}(x + B_j)$, this implies that $m^*(\Omega) = m^*(x + \Omega)$.

7.2.2 First suppose $0 < m^*_n(A) + m^*_m(B) < +\infty$, let $0 < \varepsilon < 2(m^*_n(A) + m^*_m(B))^2$, $(B_i)_{i \in I}, (B_j)_{j \in J}$ countable box coverings of A, B respectively s.t. $\sum_{i \in I} vol(B_i) < m^*_n(A) + \varepsilon'$, $\sum_{j \in J} vol(B_j) < m^*_m(B) + \varepsilon'$ where we define $\varepsilon' = \frac{\varepsilon}{2(m^*_n(A) + m^*_m(B))}$. Then, consider the covering $(B_{ij})_{i \in I, j \in J}$, where $B_{ij} = B_i \times B_j$ of $A \times B$ (Note that $B_i \times B_j$ is a box because if $B_i = \prod_{k=1}^{n} (a_k^i, b_k^i)$, $B_j = \prod_{k=1}^{m} (a_k^j, b_k^j)$,
$B_i \times B_j = \{(x_i, x_j) \mid x_i \in B_i, x_j \in B_j\} =$
$\{(x_1, \dots x_n, y_1, \dots y_m) \in \mathbb{R}^{n+m}, x_k \in (a_k^i, b_k^i) \ \forall\ 1 \le k \le n,$
$y_k \in (a_k^j, b_k^j) \ \forall\ 1 \le k \le m\}$, and $(B_{ij})_{i \in I, j \in J}$ covers $A \times B$ because for $(a, b) \in A \times B$, $a \in B_i$, $b \in B_j$ for some $i \in I$, $j \in J$).

Now, note that $Vol(B_{ij}) = Vol(B_i) Vol(B_j)$, so $\sum_{i \in I} \sum_{j \in J} Vol(B_{ij}) = \sum_{i \in I} Vol(B_i) \sum_{j \in J} Vol(B_j) <$
$(m^*_n(A) + \varepsilon')(m^*_m(B) + \varepsilon') = m^*_n(A) m^*_m(B) +$
$\varepsilon'(m^*_n(A) + m^*_m(B)) + (\varepsilon')^2 = \frac{\varepsilon}{2} + \frac{\varepsilon^2}{4(m^*_n(A) + m^*_m(B))^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$
since $\varepsilon < 2(m^*_n(A) + m^*_m(B))^2$ implies $\frac{\varepsilon^2}{4(m^*_n(A) + m^*_m(B))} < \frac{\varepsilon}{2}$.
This shows that $m^*_{n+m}(A \times B) \le m^*_n(A) m^*_m(B)$.

If $m^*(A) + m^*(B) = 0$, $m^*(A) = m^*(B) = 0$, so we may pick $\varepsilon' = \sqrt{\varepsilon}$ and repeat the same argument, with the last step becoming $\sum_{i \in I} \sum_{j \in J} Vol(B_{ij})$ $< (m^*_n(A) + \varepsilon')(m^*_m(B) + \varepsilon') = (\varepsilon')^2 = \varepsilon$.

If $m^*(A) + m^*(B) = +\infty$, either $m^*(A) = +\infty$ or $m^*(B) = +\infty$, so $m^*(A) m^*(B) = +\infty$. The result follows from positivity.

7.2.3 a) Let $E_1 = A_1$ and $E_i = A_i \setminus A_{i-1}$ for $i \geq 2$. Note that by complementarity and the Boolean Algebra property, $E_i = A_i \cap \mathbb{R}^n \setminus A_{i-1}$ is measurable. Furthermore, $\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} E_j$ and the E_j are disjoint. By countable additivity, $m(\bigcup_{j=1}^{\infty} A_j) = m(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} m(E_j)$ $= \lim_{n\to\infty} \sum_{j=1}^{n} m(E_j) = \lim_{n\to\infty} m(\bigcup_{j=1}^{n} E_j) = \lim_{n\to\infty} m(A_j)$ as desired.

b) Let $E_i = A_1 \setminus A_i$ for $i \geq 1$. Then, note that $\bigcap_{j=1}^{\infty} A_j = A_1 \setminus \bigcup_{j=1}^{\infty} E_j$, and $E_i \subseteq E_{i+1}$ for each i. By a), $m(\bigcup_{j=1}^{\infty} E_j) = \lim_{j\to\infty} m(E_j)$ $= \lim_{j\to\infty} m(A_1 \setminus A_j) = \lim_{n\to\infty} m(A_1) - m(A_j) =$ $m(A_1) - \lim_{j\to\infty} m(A_j)$, since by countable additivity $m(A_1) = m(A_1 \setminus A_j) + m(A_j)$. Similarly, $m(\bigcap_{j=1}^{\infty} A_j) = m(A_1 \setminus \bigcup_{j=1}^{\infty} E_j) =$ $m(A_1) - (m(A_1) - \lim_{j\to\infty} m(A_j)) = \lim_{j\to\infty} m(A_j)$ as desired.

7.2.4 Consider the q^n translates of $(0, 1/q)^n$ $(0, 1/q)^n + \bar{x}$; where $\bar{x} = (x_1, \dots x_n)$ and $x_i = \frac{k}{q}$ for some $k \in [0, q-1] \cap \mathbb{Z}$. Note that these translates are disjoint and by translation invariance have measure $m((0, 1/q)^n)$. Furthermore, each translate is contained in $(0,1)^n$. Let A denote the union of the translates. By countable additivity and monotonicity,

$$m(A) = q^n\, m((0, 1/q)^n) \leq m((0,1)^n) \leq m([0,1]^n) = 1$$

so $m((0, 1/q)^n) \leq q^{-n}$.

Similarly, consider the q^n translates of $[0, 1/q]^n$ $[0, \frac{1}{q}]^n + \bar{x}$, where $\bar{x} = (x_1, \dots x_n)$ and $x_i = \frac{k}{q}$ for some $k \in [0, q-1] \cap \mathbb{Z}$. By translation invariance, these translates have measure $m([0, 1/q]^n)$. Note that their union is $[0,1]^n$, so by countable sub-additivity,

$$1 = m([0,1]^n) \leq q^n\, m([0, 1/q]^n)$$

so $q^{-n} \leq m([0, 1/q]^n)$.

Now, let $\varepsilon > 0$. For each boundary piece

$D_k = \{(x_1, \dots a_i, \dots x_n) \in \mathbb{R} \mid 0 \leq x_j \leq 1/q \ \forall\, j \neq i$ and, $a_i \in \{0, 1/q\}$ constant$\}$

We can cover using the box
$$B_k = (a_i - \varepsilon', a_i + \varepsilon') \prod_{j=1, j\neq i}^{n} (-\varepsilon', 1/q + \varepsilon')$$

for any $\varepsilon' > 0$. Note that $\mathrm{vol}(B_k) = 2\varepsilon'(\frac{1}{q} + 2\varepsilon')^{n-1}$ so $\sum_k \mathrm{vol}(B_k) = 4n\,\varepsilon'(\frac{1}{q} + 2\varepsilon')^{n-1}$, as there are $2n$ boundary pieces making up $[0, 1/q]^n \setminus (0, 1/q)^n$.

Thus since $\lim_{\varepsilon' \to 0^+} 4n\,\varepsilon'\left(\frac{1}{q}+2\varepsilon'\right)^{n-1} = 0$, we may pick $\hat{\varepsilon} > 0$ s.t. $4n\hat{\varepsilon}\left(\frac{1}{q}+2\hat{\varepsilon}\right)^{n-1} < \varepsilon$. We have therefore found a covering of $[0, 1/q]^n \setminus (0, 1/q)^n$ with total volume $< \varepsilon$, and thus $m([0, 1/q]^n \setminus (0, 1/q)^n) < \varepsilon$. Since ε was arbitrary, $m([0, 1/q]^n \setminus (0, 1/q)^n) = 0$.

Putting this all together, by additivity

$$
\begin{aligned}
q^{-n} &\leq m([0, 1/q]^n) \\
&= m([0, 1/q]^n \setminus (0, 1/q)^n) + m((0, 1/q)^n) \\
&= m((0, 1/q)^n) \\
&\leq q^{-n}
\end{aligned}
$$

So $m([0, 1/q]^n) = m((0, 1/q)^n) = q^{-n}$.

7.4.1 Let $A = (a,b)$. We have $m^*(A) = b-a$.

If $(a,b) \subseteq (0,\infty)$,

$m^*(A \cap (0,\infty)) + m^*(A \setminus (0,\infty)) =$

$m^*(A) + m^*(\emptyset) = m^*(A)$

If $(a,b) \subseteq (-\infty, 0]$; $A \setminus (0,\infty) = A$ so

$m^*(A \cap (0,\infty)) + m^*(A \setminus (0,\infty)) =$

$m^*(\emptyset) + m^*(A) = m^*(A)$

If $0 \in (a,b)$,

$m^*(A \cap (0,\infty)) + m^*(A \setminus (0,\infty)) =$

$m^*((0,b)) + m^*((a,0]) =$ Note

$b + (0-a) = b-a \stackrel{?}{=} m^*(A)$

Since all cases are covered we are done.

7.4.2 If $A \cap E = A$ and $A \setminus E = \emptyset$ or $A \cap E = \emptyset$ and $A \setminus E = A$, then

$m^*(A) = m^*(A) + m^*(\emptyset)$

and we are done.

Otherwise, note that if $A = \prod_{i=1}^{n} (a_i, b_i)$,

$A \cap E = \prod_{i=1}^{n-1} (a_i, b_i) \cdot (0, b_n)$ and $A \setminus E = \prod_{i=1}^{n-1} (a_i, b_i)\,(a_n, 0]$

which are both boxes, so we can compute the measure directly to obtain

$m^*(A \cap E) + m^*(A \setminus E) =$

$b_n \prod_{i=1}^{n-1} (b_i - a_i) - a_n \prod_{i=1}^{n-1} (b_i - a_i) =$

$\prod_{i=1}^{n} (b_i - a_i) = m^*(A)$

as desired.

7.4.3 Let $A \subseteq \mathbb{R}^n$. Let $\varepsilon > 0$. We may find a countable covering $(B_j)_{j \in J}$ s.t. $\sum_{j \in J} \text{vol}(B_j) < m^*(A) + \varepsilon$. For each $j \in J$, note by 7.4.2

$$m^*(B_j) = m^*(B_j \cap E) + m^*(B_j \setminus E)$$

Since $A \cap E \subseteq \bigcup_{j \in J} (B_j \cap E)$, by monotonicity + subadditivity

$$m^*(A \cap E) \leq m^*\left(\bigcup_{j \in J} (B_j \cap E)\right) \leq \sum_{j \in J} m^*(B_j \cap E)$$

Similarly, since $A \setminus E \subseteq \bigcup_{j \in J} B_j \setminus E$,

$$m^*(A \setminus E) \leq m^*\left(\bigcup_{j \in J} B_j \setminus E\right) \leq \sum_{j \in J} m^*(B_j \setminus E)$$

Thus,

$$\begin{aligned} m^*(A \cap E) + m^*(A \setminus E) &\leq \sum_{j \in J} \left[m^*(B_j \cap E) + m^*(B_j \setminus E) \right] \\ &= \sum_{j \in J} m^*(B_j) \\ &= \sum_{j \in J} \text{vol}(B_j) \\ &< m^*(A) + \varepsilon \end{aligned}$$

So $m^*(A \cap E) + m^*(A \setminus E) \leq m^*(A)$.

For the reverse inequality, note that $A = (A \cap E) \cup (A \setminus E)$, so the result follows from sub-additivity.

7.4.4 a) Note that for $A \subseteq \mathbb{R}^n$,
$m^*(A \cap E^c) + m^*(A \setminus E^c) =$
$m^*(A \setminus E) + m^*(A \cap E) =$
$m^*(A)$
by measurability of E.

b) Let $A \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$. By measurability of E,

$$m^*(A-x) = m^*((A-x) \cap E) + m^*((A-x) \setminus E)$$
$$= m^*(A \cap (x+E)) + m^*(A \setminus (x+E))$$

but since $m^*(A-x) = m^*(A)$ by translation invariance, this shows $x+E$ is measurable.

$m(x+E) = m(E)$ follows directly from translation invariance of m^*.

c) Define

$A_{++} = A \cap E_1 \cap E_2$ $\quad$ $A_{+-} = A \cap E_1 \cap E_2^c$
$A_{-+} = A \cap E_1^c \cap E_2$ $\quad$ $A_{--} = A \cap E_1^c \cap E_2^c$.

Note that $A = A_{++} \sqcup A_{+-} \sqcup A_{-+} \sqcup A_{--}$

Since E_1 is measurable,

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$
$$= m^*(A_{++} \cup A_{+-}) + m^*(A_{-+} \cup A_{--})$$

Since E_2 is measurable,

$$m^*(A_{++} \cup A_{+-}) = m^*((A_{++} \cup A_{+-}) \cap E_2) + m^*((A_{++} \cup A_{+-}) \setminus E_2)$$
$$= m^*(A_{++}) + m^*(A_{+-})$$

$$m^*(A_{-+} \cup A_{--}) = m^*((A_{-+} \cup A_{--}) \cap E_2) + m^*((A_{-+} \cup A_{--}) \setminus E_2)$$
$$= m^*(A_{-+}) + m^*(A_{--})$$

Thus,
$$m^*(A) = m^*(A_{++}) + m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})$$

Furthermore, note that by similar logic, (letting $A' = A_{+-} \cup A_{-+} \cup A_{--}$,
$$m^*(A') = m^*(A' \cap E_1) + m^*(A' \setminus E_1)$$
$$= m^*(A_{+-}) + m^*(A_{-+} \cup A_{--})$$
$$= m^*(A_{+-} \cap E_2) + m^*(A_{+-} \setminus E_2) + m^*((A_{-+} \cup A_{--}) \cap E_2)$$
$$+ m^*((A_{-+} \cup A_{--}) \setminus E_2)$$
$$= m^*(A_{+-}) + m^*(\emptyset) + m^*(A_{--}) + m^*(A_{-+})$$
$$= m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})$$

Since $A' = A \setminus (E_1 \cap E_2)$ and $A_{++} = A \cap (E_1 \cap E_2)$, this shows $m^*(A) = m^*(A \cap (E_1 \cap E_2)) + m^*(A \setminus (E_1 \cap E_2))$.

For the union, note that
$$A \cap (E_1 \cup E_2) = A_{++} \cup A_{+-} \cup A_{-+}$$
$$A \setminus (E_1 \cup E_2) = A_{--}$$

By the same logic as before, (letting $\tilde{A} = A_{++} \cup A_{+-} \cup A_{-+}$,
$$m^*(\tilde{A}) = m^*(\tilde{A} \cap E_1) + m^*(\tilde{A} \setminus E_1)$$
$$= m^*(A_{++} \cup A_{+-}) + m^*(A_{-+})$$
$$= m^*((A_{++} \cup A_{+-}) \cap E_2) + m^*((A_{++} \cup A_{+-}) \setminus E_2) + m^*(A_{-+})$$

$= m^*(A_{+-}) + m^*(A_{++}) + m^*(A_{-+})$

so using the previously derived equality,

$m^*(A) = m^*(A_{--}) + m^*(A')$

as desired.

d) We show this by induction. For the base case $N=2$, this was proven in c.

Now, suppose the result holds for n. For the $n+1$ case, we can write

$$\bigcup_{j=1}^{n+1} = \left(\bigcup_{j=1}^{n} E_j\right) \cup E_{n+1}$$

$$\bigcap_{j=1}^{n+1} = \left(\bigcap_{j=1}^{n} E_j\right) \cap E_{n+1}$$

By the induction hypothesis, $(\bigcup_{j=1}^{n} E_j)$ and $(\bigcap_{j=1}^{n} E_j)$ are measurable, so applying part c) yields the result.

e) Consider the box $B = \prod_{i=1}^{n} (a_i, b_i)$ (parentheses not necessarily all open). Note that we can write $B = \bigcap_{j=1}^{n} A_j \cap \bigcap_{j=1}^{n} B_j$, where

$$A_j = \{ \bar{x} \in \mathbb{R}^n \mid x_j > a_j \}$$
$$B_j = \{ \bar{x} \in \mathbb{R}^n \mid x_j < b_j \}$$

(the $<$ or $>$ may be $\leq$ or $\geq$ as needed).
Note that the A_j, B_j are translates of

half-spaces or their compliments, so the result follows from a), b), d).

f) Let $A \subseteq \mathbb{R}^n$. By monotonicity,
$m^*(A \cap E) \leq m^*(E) = 0$

So $m^*(A \cap E) = 0$.

Also by monotonicity,
$m^*(A \setminus E) \leq m^*(A)$

So $m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E)$

By subadditivity, since $(A \cap E) \cup (A \setminus E) = A$
$m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E)$

Showing the desired.