HW4

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The difference between Riemann integral and Lebesgue Integral.
In terms of definition

1. Riemman: Defined when the lower Darboux sum of all possible partition is equal to the upper Darboux sum of all possible partition. In Ross, it is defined as
$$\int f = U(f) = L(f)$$
.

2. Lebesgue: Defined to be the measure of the undergraph of a function. That is
$$\int f = m(\mathcal{U}f)$$

Similarities

- both are defined in terms of the area of a function.
- continuous functions integrable in both cases

Differences

- Riemann integral is defined as sum of small vertical strip while Lebesgue integral is defined as the outer measure of the under graph, in other words, Lebesgue integral approximates the area by boxes.
- since a vertical strip can be viewed as a box, let the components of the lower Darboux sum be viewed as closed boxes, let the components of the the upper Darboux sum be viewed as open boxes (claim: differ from the $G_\delta-$set by a measure zero boundary), then the union of boxes in the lower darboux sum can be viewed as the $F_\sigma-$set of the undergraph, while the union of boxes in the upper darboux sum approximate the $G_\delta-$set of the undergraph.
- Every Riemann Integrable function is Lebesgue Integrable. (To be verified)

25. Let $f : \mathbb{R} \to [0, \infty)$ be given.

(a) If f is measurable why is the graph of f a zero set?

(b) If the graph of f is a zero set does it follow that f is measurable?

**(c) Read about transfinite induction and go to stackexchange to see that there exists a nonmeasurable function $f : [a, b] \to [0, \infty)$ whose graph is nonmeasurable.

(d) Infer that the measurability hypothesis in the Zero Slice Theorem (Theorem 26) is necessary since every vertical slice graph of the function in (c) is a zero set (it is just a single point) and yet the graph has positive outer measure.

(e) Why can a graph never have positive inner measure?

(f) How does (c) yield an example of uncountably many disjoint subsets of the plane, each with infinite outer measure?

(g) What assertion can you make from (f) and Exercise 19?

(a) the graph f is a zero set by the zero slice theorem. Since for every $x \in \mathbf{R}$, is a measure zero set in $\mathbf{R}$. Thus no x has a slice with measure not equal to zero. Thus by the zero slice theorem, the graph $\{(x, y) \in \mathbf{R}^2 : x \in \mathbf{R}, y = f(x)\}$ has measure zero, and thus is measurable.

(b) Answer with (c)

(c) Try to find a nonmeasurable function with graph measure 0 to contradict the statement. Consider the following construction: Let $A \subset \mathbf{R}$ be nonmeasurable. Define the indicator

$$\mathbb{1}_{x \in A} = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}$$

Then The indicator function is nonmeasurable by the stronger definition of measurable functions, but the graph is measure zero.
Wiki: https://en.wikipedia.org/wiki/Measurable_function
#Non-measurable_functions.
Also found this:
Stackexchange:
https://math.stackexchange.com/questions/35606/
lebesgue-measure-of-the-graph-of-a-function

(d) Dropping the measurablility assumption of the set. Suppose the slices that are non measure zero is measure zero. Consider a function f, and suppose the graph of f is nonmeasurable. Then the measure of the graph is non zero. But this contradicts the conclusion of the zero slice theorem.

(e) Suppose a graph, G, has positive inner measure. That is, there exist some F_σ−set $F \subset G$ such that $m_*(G) = m(F) > 0$. But $m(G) = 0$, thus by monotonicity, $0 \leq m(F) \leq 0$, $m(F) = 0$, a contradiction. Thus, a graph can never have positive inner measure. (assumes graph has zero measure.)

Alternative proof from stackexchange: Outline: If a function graph G of positive inner measure existed, then choosing $K \subset G$ measurable of positive measure and considering the translates of K, you would get uncountably

many disjoint measurable sets of positive measure. This is impossible in any $\sigma-$finite measure space.
https://math.stackexchange.com/questions/1899368/the-graph-of-every-real-function-has-inner-measure-zero

Possibly related. https://math.stackexchange.com/questions/1267224/disjoint-union-of-uncountable-measurable-sets-of-positive-measure

(f) Consider translates of the function considered in (c). Let $f_r : x \mapsto \mathbb{1}_{x\in A}(x)+r$, $r \in (0,1)$. The plane $\mathbf{R}^2$ is divided into disjoint (need to verify), uncountable, infinite measure (need to verify).

(g)

28. The **total undergraph** of $f : \mathbb{R} \to \mathbb{R}$ is $\underline{\mathcal{U}}f = \{(x, y) : y < f(x)\}$.

(a) Using undergraph pictures, show that the total undergraph is measurable if and only if the positive and negative parts of f are measurable.

(b) Suppose that $f : \mathbb{R} \to (0, \infty)$ is measurable. Prove that $1/f$ is measurable. [Hint: The diffeomorphism $T : (x, y) \mapsto (x, 1/y)$ sends $\mathcal{U}f$ to $\mathcal{U}(1/f)$.]

(c) Suppose that $f, g : \mathbb{R} \to (0, \infty)$ are measurable. Prove that $f \cdot g$ is measurable. [Hint: $T : (x, y) \mapsto (x, \log y)$ sends $\mathcal{U}f$ and $\mathcal{U}g$ to $\mathcal{U}(\log f)$ and $\mathcal{U}(\log g)$. How does this imply $\log fg$ is measurable, and how does use of $T^{-1} : (x, y) \mapsto (x, e^y)$ complete the proof?]

(d) Remove the hypotheses in (a)-(c) that the domain of f, g is $\mathbb{R}$.

(e) Generalize (c) to the case that f, g have both signs.

(a) Let $A = \{(x, y) \in \mathbf{R} \times [0, \infty) : y < f(x), f(x) \geq 0\}$ be the total undergraph for positive f ,and $B = \{(x, y) \in \mathbf{R} \times [0, \infty) : y < f(x), f(x) < 0\}$ be the total undergraph for negative f. Then $\underline{\mathcal{U}}f = A \cup B$, thus if A and B are measurable, then $\underline{\mathcal{U}}f$ is measurable. Conversely, if $\underline{\mathcal{U}}f$ is measurable, let the half plane be $E = \{(x, y) \in \mathbf{R}^2 | y > 0\}$, then $E^c = \{(x, y) \in \mathbf{R}^2 | y \leq 0\}$. Since $E, E^c, \underline{\mathcal{U}}f$ are all measurable, thus, $\underline{\mathcal{U}}f \cap E$ and $\underline{\mathcal{U}}f \cap E^c$ are both measurable.

(b) Want to show that $\mathcal{U}(1/f)$ is measurable. Let $(x, y) \in \mathcal{U}f$, then $y < f(x)$, then $1/y > 1/f(x)$. Since $T(x, y) = (x, 1/y) \in (\widehat{\mathcal{U}}(1/f))^c$, thus $T : \mathcal{U}f \to (\widehat{\mathcal{U}}(1/f))^c$. And since T is a diffeomorphism, thus a meseomorphism, and thus $(\hat{\mathcal{U}}(1/f))^c$ is measurable, and so $\hat{\mathcal{U}}(1/f)$ is measurable, and so is $\mathcal{U}(1/f)$ (since the boundary is a measure zero set).

(c) Want to show that $\mathcal{U}fg$ is measurable. Let f, g be measurable functions, then $\mathcal{U}f$ and $\mathcal{U}g$ are measurable. Let $T_1, T_2 : (x, y) \mapsto (x, \ln y)$ are diffeomorphisms, and $T_1 : \mathcal{U}f \to \mathcal{U}\ln f$, $T_2 : \mathcal{U}f \to \mathcal{U}\ln f$, thus $\mathcal{U}\ln f$ and $\mathcal{U}\ln g$ are measurable. And so

$$\begin{aligned}
\mathcal{U}\ln f + \mathcal{U}\ln g &= \int \ln f + \int \ln g \\
&= \int \ln f + \ln g \\
&= \mathcal{U}(\ln f + \ln g) \\
&= \mathcal{U}(\ln f \cdot g)
\end{aligned}$$

$\mathcal{U}(\ln f \cdot g)$ is measurable. Then let $T_3 : (x, y) \mapsto (x, e^y)$, is a diffeomorphism, and so a meseomorphism, and since $T_3 : \mathcal{U}(\ln f \cdot g) \to \mathcal{U}(f \cdot g)$, thus $\mathcal{U}(f \cdot g)$ is a measurable.

(d) for the scenarios (a),(b),(c), since the methods don't depend on the fact that the domain is in $\mathbf{R}$, thus all the conclusions still hold.

(e) By Tao's definition of measurable function, continuous functions are measurable, $h : x \mapsto x^2$ is continuous, thus measurable. Thus $(h \circ (f + g) - h \circ (f - g))/4 = f \cdot g$ is measurable.

Method 2: Want to show that $\mathcal{U}(fg)$ is measurable. Let $(x, y) \in \mathcal{U}(fg)$, then $(x, y) \in \mathcal{U}(f^+g^+) \cup \mathcal{U}(f^-g^+) \cup \mathcal{U}(f^+g^-) \cup \mathcal{U}(f^-g^-)$. We know that $\mathcal{U}(f^+g^+)$ is measurable.

Since $T : (x, y) \mapsto (x, -y)$ is a mesometry, thus suppose $\mathcal{U}(f^-)$ is measurable, so is $\widehat{\mathcal{U}}(-f^-)^c$, and so is $\mathcal{U}(-f^-)$, similarly, we can get $\mathcal{U}(-g^-)$ is measurable. And both the range of $-g^-$ and $-f^-$ are non-negative, thus we could apply the result of (c). thus $\mathcal{U}(-f^- \cdot -g^-) = \mathcal{U}f^- \cdot g^-$ is measurable.

Similarly, we could do likewise for $\mathcal{U}(f^+ \cdot g^-)$, first get that $\mathbb{U}(f^+ \cdot -g^-)$ is measurable, then get $\mathcal{U}(f^+ \cdot g^-)$ is measurable by reflection.

And so $\mathcal{U}(f^+g^+) \cup \mathcal{U}(f^-g^+) \cup \mathcal{U}(f^+g^-) \cup \mathcal{U}(f^-g^-)$ is measurable by the $\sigma-$algebra property.