HW3

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Problem 1

Ex 3. Consider the special case of a diagonal line $\{y = x\}$ in $\mathbb{R}^2$. Can you prove directly that it has measure zero, i.e, using covering by boxes? You then may want to read Section 6.3 Theorem 9. Or, can you use today's zero slice theorem to prove it?

a. I don't think we could cover the non parallel lines with boxes, for a diagonal line, the the most efficient covering would be by squares (equal height and length, say l), however, there is no way to set the length l such that the sum of volume of squares $\lim_{n\to\infty} \sum_{i=1}^{n} l^2 < \epsilon$ for any $\epsilon > 0$

Proof. (for lines) We apply the measure zero theorem. Since the function $E = \{(x, y) : y = x\}$ is one to one, thus for every slice $E_x = \{y : y = x\}$ is a singleton. thus for every $x \in \mathbb{R}$, $m^*(E_x) = m^*(\{x\}) = 0$. And so the set $Z = \{x \in \mathbb{R} : m^*(E_x) \neq 0\} = \emptyset$, thus $m^*(Z) = 0$ by the property of outer measure. And so the set $E = \{(x, y) : x = y\}$ is measure zero by the zero slice theorem. □

Proof. (for higher dimensions) Similarly, for higher dimensional setting, let $E \subset \mathbf{R}^{n+k}$ be a hyper plane not parallel to the coordinate axes $\mathbf{R}^n$, let $x \in \mathbf{R}^n$. Define the slice map $E_x : \mathbf{R}^n \to \mathbf{R}^k$ that maps $x \in \mathbf{R}^n$ to $y \in \mathbf{R}^k$, is a bijection. Thus E_x is a hyperplane of $\mathbf{R}^k$ (or a singleton in $\mathbf{R}^k$), thus measure zero. This is true for all x. Thus $Z = \{x \in \mathbf{R}^n : m^*(E_x) \neq 0\} = \emptyset$, thus $m^*(E)$ is zero.

Not sure how the high dimensional problem is defined. And some claim seems to be intuitively true but have not yet verified. □

Problem 2

Ex 6. How to reduce the unbounded case to bounded case?

Theorem 16

Theorem Lebesgue measure is regular in the sense that each measurable set E can be sandwiched between an F_σ-set and a G_δ-set, $F \subset E \subset G$, such that $m(G \setminus F) = 0$. Conversely, if there is such an $F \subset E \subset G$ then E is measurable.

Proof. $\Leftarrow$ Suppose that there is $F \subset E \subset G$. We want to show that E is measurable. It suffices to show that E is the union of a zero set and a measurable set. Let G and F be the G_δ-set and F_σ-set of E. Since $G \setminus F$ is a zero set, $E \cap (G \setminus F)$ is also a zero set. F is the countable union of closed sets and thus measurable. And $E = (E \cap (G \setminus F)) \cup F$ thus E is measurable.

$\Rightarrow$ Suppose that E is measurable, unbounded. For every $n > 0$ there exists G_k open cover such that $m^*(G_n \setminus E) \leq \frac{1}{n}$. Let $G = \bigcap G_n$. Then G is also a $G_\delta-$set. And by monotonicity $m^*(G \setminus E) \leq m^*(G_n \setminus E) \leq \frac{1}{n}$. So, $m^*(G \setminus E) = 0$.

Similarly, $\mathbb{R}^n \setminus E$ is measurable, so for every $n > 0$ there exists open cover H_n such that $m^*(H_n \setminus (\mathbb{R}^n \setminus E)) < \frac{1}{n}$. Let $H = \bigcap H_n$, then we get $m^*(H \setminus (\mathbb{R}^n \setminus E)) = 0$ Let $F = R \setminus H$, F is a $F_\sigma-$set, and $m^*(A \setminus F) = m^*(H \setminus (\mathbb{R}^n \setminus E)) = 0$

Since $m^*(A \setminus F) = 0 = m^*(A) - m^*(F)$, $m^*(G \setminus A) = m^*(G) - m^*(A)$, thus $m^*(G \setminus F) = m^*(G) - m^*(A) + m^*(A) - m^*(F) = 0$

□

Theorem 16 (WRONG QUESTION)

Every open set in $n-$space is a countable disjoint union of open cubes plus a zero set.

Proof. **Route 1** $\because$ every open set can be written as union of countable open boxes, $\therefore$ it suffices to show that the union of open boxes can be written as a countable disjoint union of open boxes plus a zero set.

Without loss of generality, we claim that any two open boxes can be partitioned into disjoint open boxes plus a measure zero set. let $B_1 = \prod_{i=1}^n (a_i, b_i) = \prod_{i=1}^n I_i$, $B_2 = \prod_{i=1}^n (a'_i, b'_i) = \prod_{i=1}^n I'_i$ for every dimension, consider let

$$l_i = \min\{|I_i \setminus I'_i|, |I'_i \setminus I_i|, |I_i \cap I'_i|\}$$

$B_1 \cap B_2 \neq \emptyset$. then each box (may be partially open or closed) can be partitioned into smaller boxes of volume $\text{vol}(B_{\text{new}}) = \prod_{i=1}^n l_i$. Now, it suffices to show that every box is the union of an open box union a measure zero set.

Now, we prove that any box is a union of an open box and a measure zero set. Let $B = \prod_{i=1}^n I_i$ be a box, for any dimension d, if any interval composing the box is closed, say $I_d = [a_d, b_d]$, then make the interval $I_d = (a_d, b_d) \cup \{a_d\} \cup \{b_d\}$. Then $B = \prod_{i \neq d}^n I_i \times (a_d, b_d) \cup \{a_d\} \cup \{b_d\} = \prod_{i \neq d}^n I_i \times (a_d, b_d) \cup Z$, where Z denotes the zero set (since hyperplanes are measure zero). Doing this iteratively for every dimension. We showed the box would be equivalent to $B = \prod_{i=1}^{n} I_i' \cup Z'$, where (I_i') are open intervals, and Z' is a zero set.

Thus every open set is the union of open boxes union a measure zero set. $\square$

Theorem 21

If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^k$ are measurable, then $A \times B$ is measurable and

$$m(A \times B) = m(A)m(B)$$

by convention, $0 \cdot \infty = 0 = \infty \cdot 0$

Proof. Let A, B be measurable, define the sequence $A_i = A \cap (-2^i, 2^i)^n$, $B_i = B \cap (-2^i, 2^i)^n$, then disjointize the new sequences into $A_i' = A_i \setminus A_{i-1}$, $B_i' = B_i \setminus B_{i-1}$. We claim each A_i' and B_i' is measurable and bounded, since A, B are measurable, and boxes are measurable and bounded, thus A_i, B_i is measurable and bounded. And so is A_i', B_i'. Thus

$$\begin{aligned} A \times B &= \bigsqcup_{i=1}^{\infty} A_i' \times \bigsqcup_{j=1}^{\infty} B_j' \\ &= \bigsqcup_{j=1}^{\infty} \bigsqcup_{i=1}^{\infty} A_i' \times B_j' \end{aligned}$$

Since $A_i' \times B_j'$ is measurable, thus the countable union $A \times B$ is also measurable. And so.

$$\begin{aligned} m(A \times B) &= m\left(\bigsqcup_{j=1}^{\infty} \bigsqcup_{i=1}^{\infty} A_i' \times B_j' \right) \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} m(A_i')m(B_i') \\ &= \sum_{j=1}^{\infty} m(A_i') \sum_{i=1}^{\infty} m(B_i') \\ &= m(A)m(B) \end{aligned}$$

$\square$

Problem 3

Ex 12. You can assume properties in Ex 11.

$$J^*A = J^*\bar{A} = m\bar{A}$$

Proof. First, we prove that $J^*A = J^*\bar{A}$ $J^*A \leq J^*\bar{A}$ by monotonicity. Thus it suffices to prove that $J^*A \geq J^*\bar{A}$, equivalent to $J^*A + \epsilon \geq J^*\bar{A}$, for any $\epsilon > 0$.

Since $\bar{A}$ is A plus boundary points. Thus we first show that the outer Jordan content singletons $\{a\}$, $a \in \mathbb{R}$ are zero. since by measure of empty set and monontonicity, we get

$$J^*\emptyset \leq J^*\{a\} \leq J^*(a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2}) \leq \epsilon$$

this is true for any $\epsilon > 0$, thus, $J^*\{a\} = 0$.

Now, let $\epsilon > 0$, there exist finite cover of A, (I_j) such that $\sum_{j=1}^{n} |I_j| \leq J^*A + \epsilon$. Then let $I'_j = I_j \cup \inf\{I_j\} \cup \sup\{I_j\}$, then the finite closed covering I'_j is a covering of A, and thus also for $\bar{A}$, and so by monotonicity

$$J^*\bar{A} \leq J^* \bigcup_{j=1}^{n} I'_j \leq \sum_{j=1}^{n} J^*I'_j = \sum_{j=1}^{n} I_j \leq J^*A + \epsilon$$

Thus

$$J^*A = J^*\bar{A}$$

Now we prove that $m^*\bar{A} = J^*\bar{A}$. Since the covers considered by $m^*\bar{A}$ is a subset of covers considered in $J^*\bar{A}$, thus by property of infimum,

$$m^*\bar{A} \leq J^*\bar{A}$$

Now it suffices to prove that $m^*\bar{A} \geq m^*\bar{J}$.

Other way around. Since the outer Jordan measure of $\bar{A}$ exists, thus, $\bar{A}$ is bounded, and since $\bar{A}$ is closed, thus $\bar{A}$ is compact. Thus, for every open cover, there exists finite subcover. Thus let $\epsilon > 0$, then there exist (B_j) countable open box cover such that $\sum_{j \in J} |B_j| \leq m^*\bar{A} + \epsilon$. There exists finite subcover B'_j that covers $\bar{A}$, so

$$J^*\bar{A} \leq \sum_{j=1}^{n} |B'_j| \leq \sum_{j \in J} |B_j| \leq m^*\bar{A} + \epsilon$$

Thus $m^*\bar{A} = J^*\bar{A}$, and so

$$J^*A = J^*\bar{A} = m\bar{A}$$

□