Math 105

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HW2 (Preliminary)

Lemma 0: $m^*(E \cup F) \le m^*(E) + m^*(F)$

Working in $\mathbb{R}^n$

Let $\{B_j\}$ be a open covering of $E \cup F$ such that each B_j is a cube in n dimensions with side length no greater than $\frac{dist(E,F)}{\sqrt{n}}$

Further, let $\sum_i |B_j| \le m^*(E \cup F) + \varepsilon$

(Probably need to prove this is possible)

$\{B_j\} = \{B_j^{(1)}\} \sqcup \{B_j^{(2)}\}$, where $\{B_j^{(1)}\}$ covers E and $\{B_j^{(1)}\} \cap F = \phi$ and $\{B_j^{(2)}\}$ covers F where $\{B_j^{(2)}\} \cap E = \phi$

$\sum_i |B_j| = \sum_i |B_j^{(1)}| + \sum |B_j^{(2)}| \ge m^*(E) + \delta_1 + m^*(F) + \delta_2$ $\forall \varepsilon > 0$

$m^*(E) + m^*(F) \le \sum |B_j^{(1)}| + \sum_i |B_j^{(2)}| = \sum_i |B_j| \le m^*(E \cup F) + \varepsilon$

possible bc disjoint union

So $m^*(E) + m^*(F) = m^*(E \cup F)$ when $dist(E,F) > 0$

Lemma 1

Let $U = \bigcup B_j$, where $\{B_j\}$ is a open covering of A by countable open boxes

$\bigcup B_j$ is open as all B_j are open. If we can show that $m^*(B_j) = |B_j|$, Then $\inf\{m^*(U) \mid U \supset A, U \text{ open}\} \le \inf\{\sum_i |B_j|, \bigcup B_j \supset A, J \text{ countable}\} = m^*(A)$, and $m^*(A) \le \inf\{m^*(U) \mid U \supset A, U \text{ open}\}$, as $A \subset \{U_1, U_2, U_3, \ldots U_k, \ldots\}$, $m^*(A) \le m^*(U_k)$, so $m^*(A)$ is a lower bound on $m^*(U_k)$, so $m^*(A) \le \inf\{m^*(U) \mid U \supset A, U \text{ open}\}$

Pf that $m^*(B_j) = |B_j|$ (Pending)

Lemma 2: If $\{E_i\}$ countable collection of measurable, then $\cup E_i$ measurable

$\forall E_i, \forall \varepsilon > 0, \exists U_i$ open s.t. $U_i \supset E_i$ and $m^*(U_i \setminus E_i) < \varepsilon$

Let $\varepsilon > 0$.

$(\cup U_i) \setminus (\cup E_i) \subseteq \cup (U_i \setminus E_i)$ as

$(\cup U_i) \cap (\cup E_i)^c = \cup (U_i \cap (\cup E_i)^c)$. $(\cup E_i)^c \subset E_i^c$, so

$\cup (U_i \cap (\cup E_i)^c) \subset \cup (U_i \cap E_i^c) = \cup (U_i \setminus E_i)$

$m^*((\cup U_i) \setminus (\cup E_i)) \leq m^*(\cup (U_i \setminus E_i)) \leq \sum_i m^*(U_i \setminus E_i)$

$= \sum_i \varepsilon_i$. As ε_i can be arbitrary small while positive, let $\varepsilon_i = \frac{\varepsilon}{2^i}$. As U_i are countable, this implies

$\sum_i \varepsilon_i = \varepsilon$, so $\forall \varepsilon > 0, \exists V$ open (U_i open $\Rightarrow \cup U_i$ open)

$V \supset \cup E_i$ s.t. $m^*(V \setminus (\cup E_i)) < \varepsilon$, where $V = \cup U_i$

Lemma 3 Every closed subset $A \subset \mathbb{R}^n$ measurable

Working in $\mathbb{R}^n$

$A = \cup (A \cap C_j)$, where $C_j = [0,1]^n + \vec{x}$, where

$\vec{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \mathbb{Z}$. This is because

$\cup (A \cap C_j) = A \cap (\cup C_j) = A \cap \mathbb{R}^n = A$

$\mathbb{R}^n \subseteq \cup C_j$ as $\forall \vec{v} \in \mathbb{R}^n$, $\vec{v} = (v_1, \ldots, v_n)$. $\exists\, m_i \in \mathbb{Z}$ s.t.

$m_i \leq v_i < m_i + 1$, so $v \in \prod [m_i, m_i + 1] = (m_1, m_2 \ldots m_n) + [0,1]^n$

$(A \cap C_j) \subseteq C_j$, and C_j is bounded ($|C_j| = 1$) so $A \cap C_j$ is bounded

A is closed and C_j is closed $\Rightarrow A \cap C_j$ is closed.

so A can be written as a union of closed + bounded sets

5(cont)

Let D be closed and bounded in $\mathbb{R}^n$

(If we show D measurable, by Lemma 2, A is measurable)

WTS $\exists U$ open, $U \supset D$ s.t. $m^*(U \setminus D) < \varepsilon$ $\forall \varepsilon > 0$

Let B be an open singular box s.t. $B \supset D$. Let $\{B_j\}$ be an open cover of D by open boxes, such that $\sum_i |B_j| \leq m^*(D) + \varepsilon$.

$(\cup B_j) \cap D^c$ is open: $\cup B_j$ is open and D^c is open, so finite intersection remains open. $(\cup B_j) \cap D^c$ is therefore measurable (take the open containing set to be itself).

$m^*((\cup B_j) \cap D^c) \sqcup (\cup B_j \cap D)) = m^*((\cup B_j) \cap D^c) + m^*(D)$
$= m^*(\cup B_j)$, so $m^*((\cup B_j) \cap D^c) < \varepsilon$, so
$m^*((\cup B_j) \setminus D) < \varepsilon$ $\forall \varepsilon > 0$
so D is measurable

Note $U^c \subset E^c$ as $U \supset E$

Lemma 4: U open, $U \supset E$ s.t. $\forall \varepsilon > 0$ $m^*(U \setminus E) < \varepsilon$.
U^c is closed, so measurable by Lemma 3. Let $\varepsilon > 0$.
Let $\{B_j\}$ be a countable cover of $U \setminus E$ s.t.

$\sum_i |B_j| \leq m^*(U \setminus E) + \varepsilon/2$. Let $m^*(U \setminus E) < \varepsilon/2$

Then AND $m^*(U^c \cup (\cup B_j) \setminus E^c) < \varepsilon$ $\forall \varepsilon \geq 0$

$U^c \cup (\cup B_j) \supset E^c$: $E^c = U^c \cup (U \cap E^c)$

$\cup B_j$ is countable and $\cup B_j$ is open, so $\cup B_j$ is measurable:
$\exists (V = \cup B_j)$ s.t. $V \supset \cup B_j$, where $m^*(V \setminus \cup B_j) = m^*(\phi) = 0 < \varepsilon$ $\forall \varepsilon > 0$ ✓
Countable union of measurable is measurable, so
E measurable $\rightarrow$ E^c measurable