HW1

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Problem 1 (Exercise 7.2.1)

Prove Lemma 7.2.5., i.e. Prove

(v)

(Empty set) The empty set $\emptyset$ has outer measure $m^*(\emptyset) = 0$.

Proof. Let $B = \prod_{i=1}^n (a_i, b_i)$, where $a_i = b_i$. Since $\emptyset \subseteq B$, thus the lower bound $m^*(\emptyset) \leq vol(B) = 0$. By (vi positivity), $m^*(\emptyset) \geq 0$, thus $m^*(\emptyset) = 0$ □

(vi)

(Positivity) We have $0 \leq m^*(\Omega) \leq +\infty$ for every measurable set Ω.

Proof. By definition let $\Omega \subseteq \mathbf{R}^n$,

$$m^*(\Omega) = \inf \left\{ \sum_{j \in J} vol(B_i) : (B_j)_{j \in J} \text{ covers } \Omega; J \text{ at most countable} \right\}$$

Since $vol(B) \geq 0$ for any open box B, thus

$$m^*(\Omega) \geq 0$$

□

(vii)

(Monotonicity) If $A \subseteq B \subseteq \mathbf{R}^n$, then $m^*(A) \leq m^*(B)$.

Proof. Any open cover $\{B_i\}$ of B is also an open cover of A, so

$$\begin{aligned}
&\{\{B_i\}|\{B_i\} \text{ is an open cover of } B\} \subseteq \{\{A_i\}|\{A_i\} \text{ is an open cover of } A\} \\
\rightarrow &\{\sum_i vol(B_i)|\{B_i\} \text{ is an open cover of } B\} \\
&\subseteq \{\sum_i vol(A_i)|\{A_i\} \text{ is an open cover of } A\} \\
\rightarrow &\inf\{\sum_i vol(A_i)|\{A_i\} \text{ is an open cover of } A\} \\
&\leq \inf\{\sum_i vol(B_i)|\{B_i\} \text{ is an open cover of } B\} \\
\rightarrow &m^*A \leq m^*B
\end{aligned}$$

□

(viii)

(Finite sub-additivity) If $(A_j)_{j\in J}$ are a finite collection of subsets of $\mathbf{R}^n$, then $m^*(\cup_{j\in J}A_j) \leq \sum_{j\in J} m^*(A_j)$

Proof. First, we try to show the case where there are only two sets, in other words,

$$m^*(A \cup B) \leq m^*(A) + m^*(B)$$

, for any $A, B \subseteq \mathbf{R}^n$. Let $\epsilon > 0$, there exist an open cover $\{A_j\}$ such that

$$\sum_j vol(A_j) \leq m^*(A) + \epsilon/10$$

Similarly, there exist an open cover $\{B_j\}$ such that

$$\sum_j vol(B_j) \leq m^*(B) + \epsilon/10$$

By construct, the set $\left(\bigcup_j B_j\right) \cup \left(\bigcup_j A_j\right)$ is an open cover of $A \cup B$, thus it follows that

$$\begin{aligned}
m^*(A \cup B) &\leq \sum_j vol(A_j) + \sum_j vol(B_j) \\
&\leq m^*(B) + m^*(A) + \epsilon/5
\end{aligned}$$

Since the equation is true for all $\epsilon > 0$, thus

$$m^*(A \cup B) \leq m^*(A) + m^*(B)$$

Now, we extend the result to finite sets by induction. For any arbitrary sets $C_1, C_2 \subseteq \mathbf{R}^n$

$$m^*(C_1 \cup C_2) \leq m^*(C_1) + m^*(C_2)$$

. Suppose

$$m^*(C_1 \cup C_2 \cup \ldots \cup C_n) \leq m^*(C_1) + m^*(C_2) + \ldots + m^*(C_n)$$

Then

$$\begin{aligned} &m^*(C_1 \cup C_2 \cup \ldots \cup C_n \cup C_{n+1}) \\ =&m^*((C_1 \cup C_2 \cup \ldots \cup C_n) \cup C_{n+1}) \\ \leq&m^*(C_1 \cup C_2 \cup \ldots \cup C_n) + m^*(C_{n+1}) \\ \leq&m^*(C_1) + m^*(C_2) + \ldots + m^*(C_n) + m^*(C_{n+1}) \end{aligned}$$

Thus by mathematical induction,

$$m^*(\cup_{j\in J} A_j) \leq \sum_{j\in J} m^*(A_j)$$

for any finite collection of sets. □

(x)

(Countable sub-additivity) If $(A_j)_{j\in J}$ are a countable collection of subsets of $\mathbf{R}^n$, then $m^*(\cup_{j\in J} A_j) \leq \sum_{j\in J} m^*(A_j)$

Proof. Let $\epsilon > 0$, there exists $(B_i^{(j)})$ such that

$$\sum_{i=1}^{\infty} vol(B_i^{(j)}) \leq m^*(A_j) + \epsilon/2^j$$

Thus,

$$\begin{aligned} m^*(\bigcup_{j=1}^{\infty} A_j) \leq& \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} vol(B_i^{(j)}) \\ \leq& \sum_{j=1}^{\infty} (m^*(A_j) + \epsilon/2^j) \\ \leq& \sum_{j=1}^{\infty} m^*(A_j) + \epsilon \end{aligned}$$

This is true for any $\epsilon > 0$, thus

$$m^*(\bigcup_{j\in J} A_j) \leq \sum_{j\in J} m^*(A_j)$$

□

(xiii)

(Translation invariance) If Ω is a subset of $\mathbf{R}^n$, and $x \in \mathbf{R}^n$, then $m^*(x+\Omega) = m^*(\Omega)$

Proof. WTS

$$m^*(x+\Omega) = m^*(\Omega)$$

which is equivalent as showing

$$m^*(x+\Omega) \geq m^*(\Omega)$$

and

$$m^*(x+\Omega) \leq m^*(\Omega)$$

Let $\epsilon > 0$, then there exist sequence of open boxes (B_j) such that

$$\sum_{j=1} vol(B_j) \leq m^*(\Omega) + \epsilon$$

but then $(B_j + x)$ is also an open cover for $\Omega + x$ thus by countable subadditivity.

$$m^*(\Omega + x) \leq \sum_j vol(B_j + x) = \sum_j vol(B_j) \leq m^*(\Omega) + \epsilon$$

This is true for all $\epsilon > 0$, thus

$$m^*(\Omega + x) \leq m^*(\Omega)$$

Similarly, there exist a sequence of covers (C_j) such that

$$\sum_{j=1} vol(C_j) \leq m^*(\Omega + x) + \epsilon$$

but then $(C_j - x)$ would be an open cover of Ω, thus by countable subadditivity

$$m^*(\Omega) \leq \sum_j vol(C_j - x) = \sum_j vol(C_j) \leq m^*(\Omega + x) + \epsilon$$

This is true for all $\epsilon > 0$, thus

$$m^*(\Omega) \leq m^*(\Omega + x)$$

and so we conclude that

$$m^*(\Omega) = m^*(\Omega + x)$$

□

Problem 2 (Exercise 7.2.2)

Let A be a subset of $\mathbf{R}^n$, and let B be a subset of $\mathbf{R}^m$. Show that $m^*_{n+m}(A \times B) \leq m^*_n(A) m^*_m(B)$

Proof. Let $A \in \mathbf{R}^n$, $B \in \mathbf{R}^m$, let $\epsilon > 0$, let $\epsilon' = \min\{\epsilon, 1, m^*(A) + m^*(B)\}$. Then we know for every $\epsilon > 0$, there exists an open cover for A, (A_j) such that

$$\sum vol(A_j) \leq m^*(A) + \frac{\epsilon'}{10(m^*(A) + m^*(B))}$$

and that there exists an open cover for B, (B_j) such that

$$\sum vol(B_j)) \leq m^*(B) + \frac{\epsilon'}{10(m^*(A) + m^*(B))}$$

And we know that $A \times B \subseteq \bigcup_i \bigcup_j A_i \times B_j$, and since (A_i), (B_j) are sequence of open boxes, thus

$$\begin{aligned}
m^*(A \times B) &\leq m^*(\bigcup_i \bigcup_j A_i \times B_j) \\
&= \sum_i \sum_j vol(A_i) vol(B_j) \\
&= \sum_i vol(A_i) \sum_i vol(B_j) \\
&\leq (m^*(B) + \frac{\epsilon'}{10(m^*(A) + m^*(B))})(m^*(A) + \frac{\epsilon'}{10(m^*(A) + m^*(B))}) \\
&= m^*(A)m^*(B) + \epsilon'/10 + \frac{\epsilon'^2}{10(m^*(A) + m^*(B))} \\
&\leq m^*(A)m^*(B) + \epsilon'/10 + \frac{\epsilon'}{10(m^*(A) + m^*(B))} \\
&< m^*(A)m^*(B) + \epsilon' \\
&\leq m^*(A)m^*(B) + \epsilon
\end{aligned}$$

This is true for all $\epsilon > 0$, thus

$$m^*_{n+m}(A \times B) \leq m^*_n(A) m^*_m(B)$$

□

Problem 3 (Exercise 7.2.3)

(a)

Show that if $A_1 \subseteq A_2 \subseteq A_3 \dots$ is an increasing sequence of measurable sets, then we have $m(\cup_{j=1}^{\infty} A_j) = \lim_{j\to\infty} m(A_j)$

Proof. WTS for $\epsilon > 0$ there exist some $N \in \mathbf{R}$ such that for $j \geq N$

$$\left(m^*(A_j) - m^*(\cup_{j=1}^{\infty} A_j)\right) < \epsilon$$

since $m^*(\cup_{j=1}^{\infty} A_j)$ is the least upper bound for all $S = \{m^*(\cup_{j=1}^{n} A_j) | n \in \mathbf{N}\}$ (this is true since suppose sup is greater than $m^*(\cup_{j=1}^{\infty} A_j)$, there would be a union of sequence of sets whose measure is greater than $m^*(\cup_{j=1}^{\infty} A_j)$ but any union of sequence of A_j would be a subset of $m^*(\cup_{j=1}^{\infty} A_j)$, which would contradict result of subadditivity, on the other hand, we know that for every $N \in \mathbf{N}$ $(\cup_{j=1}^{N} A_j) \subset (\cup_{j=1}^{\infty} A_j)$, thus $m^*(\cup_{j=1}^{N} A_j) \leq m^*(\cup_{j=1}^{\infty} A_j)$, $m^*(\cup_{j=1}^{\infty} A_j)$ is an upper bound), thus there exist $\cup_{j=1}^{n} A_j \in S$ such that

$$m^*(\cup_{j=1}^{\infty} A_j) - \epsilon/2 \leq m^*(\cup_{j=1}^{n} A_j) \leq m^*(\cup_{j=1}^{\infty} A_j)$$

and since the sequence $m^*(\cup_{j=1}^{n} A_j)$ is monontone increasing, thus

$$m^*(\cup_{j=1}^{\infty} A_j) - \epsilon/2 \leq m^*(\cup_{j=1}^{m} A_j) \leq m^*(\cup_{j=1}^{\infty} A_j)$$

for all $m \geq n$, thus choose $N = n$ and since $\cup_{j=1}^{m} A_j = A_j$ thus

$$\begin{aligned} &m^*(\cup_{j=1}^{\infty} A_m) - \epsilon/2 \leq m^*(A_j) \leq m^*(\cup_{j=1}^{\infty} A_m) \\ &\to |m^*(A_j) - m^*(\cup_{j=1}^{\infty} A_j)| < \epsilon \end{aligned}$$

So $m(\cup_{j=1}^{\infty} A_j) = \lim_{j\to\infty} m(A_j)$ □

(b)

Show that if $A_1 \supseteq A_2 \supseteq A_3 \dots$ is a decreasing sequence of measurable sets, and $m(A_1) < +\infty$, then we have $m(\cap_{j=1}^{\infty} A_j) = \lim_{j\to\infty} m(A_j)$

Proof. WTS that for every $\epsilon > 0$, there exist $N \in \mathbf{R}$ such that for every $\epsilon > 0$, there exist $j \in \mathbf{N}$ such that if $j > N$

$$|m(A_j) - m(\cap_{j=1}^{\infty} A_j)| < \epsilon$$

□

Since $m(\cap_{j=1}^{\infty} A_j)$ is the inf of the set $S = \{m^*(\cup_{j=1}^{N} A_j) | N \in \mathbf{N}\}$ (since suppose the inf $S < m(\cap_{j=1}^{\infty} A_j)$), then there is some $m(\cap_{j=1}^{n} A_j) < m(\cap_{j=1}^{\infty} A_j)$ but then $(\cap_{j=1}^{n} A_j) \subseteq (\cap_{j=1}^{n} A_j)$ but by subadditivity, a contradiction.) Thus there exist N such that

$$m(\cap_{j=1}^{\infty} A_j) \leq m(\cap_{j=1}^{N} A_j) \leq m(\cap_{j=1}^{\infty} A_j) + \epsilon/2$$

since $m(\cap_{j=1}^N A_j)$ is monotone decreasing with respect to N, thus for all $n > N$

$$m(\cap_{j=1}^\infty A_j) \leq m(\cap_{j=1}^n A_j) \leq m(\cap_{j=1}^N A_j) \leq m(\cap_{j=1}^\infty A_j) + \epsilon/2$$

$$\to m(\cap_{j=1}^\infty A_j) \leq m(A_n) \leq m(\cap_{j=1}^N A_j) \leq m(A_N) + \epsilon/2$$

$$\to |m(A_n) - m(\cap_{j=1}^\infty A_j)| < \epsilon$$

Thus it follows that

$$m(\cap_{j=1}^\infty A_j) = \lim_{j\to\infty} m(A_j)$$

Problem 4 (Exercise 7.2.4)
will revise, assumed outer measure

Show that for any positive integer $q > 1$, that the open box

$$(0, 1/q)^n := \{(x_1, \dots, x_n) \in \mathbf{R}^n : 0 < x_j < 1/q \text{ for all } 1 \leq j \leq n\}$$

and the closed box

$$[0, 1/q]^n := \{(x_1, \dots, x_n) \in \mathbf{R}^n : 0 \leq x_j \leq 1/q \text{ for all } 1 \leq j \leq n\}.$$

both measure q^{-n}

Proof. First we prove that $m^*((0, 1/q)^n) \leq 1/q^n$. We prove the claim by mathematical induction. In $\mathbf{R}$, $m^*((0, 1/q)) = 1/q \leq 1/q$. Now suppose $m^*((0, 1/q)^n) \leq 1/q^n$, then

$$\begin{aligned} m^*((0, 1/q)^{n+1}) &= m^*_{n+1}((0, 1/q)^n \times (0, 1/q)) \\ &\leq m^*_n((0, 1/q)^n) * m^*_n((0, 1/q)) \\ &\leq 1/q^n \cdot 1/q = 1/q^{n+1} \end{aligned}$$

Thus $m^*((0, 1/q)^n) \leq 1/q^n$ for all $n \in \mathbf{N}$.

Now we prove that $m^*([0, 1/q]^n) \geq 1/q^n$. First we cover $[0, 1]^n$ by

$$S = \{[0, 1/q]^n + 1/q(l_1, l_2, \dots, l_n) | l_n \in \{0, 1, 2, \dots, q-1\}$$

, $\bigcup_{S_i \in S}^{q^n} Si = [0, 1]$ then

$$\begin{aligned} m^*([0, 1]) &= m^*(\bigcup_{S_i \in S}^{q^n} Si) \\ &\leq \sum_{i=1}^{q^n} m^*([1, 1/q]) \\ &= q^n m^*([1, 1/q]) \\ \to m^*([1, 1/q]) &\geq 1/q^n \end{aligned}$$

Now we prove that $m([0, 1/q]^n \setminus (0, 1/q)^n) \leq \epsilon$ for every $\epsilon > 0$. Let

$$A_j = \{x \in \mathbf{R}^n | q - \frac{q^{n-1}}{8n}\epsilon \leq x_j \leq q + \frac{q^{n-1}}{8n}\epsilon, \text{for } k \neq j, 0 < x_k < 1/q\}$$

$$B_j = \{x \in \mathbf{R}^n | 0 - \frac{q^{n-1}}{8n}\epsilon \leq x_j \leq 0 + \frac{q^{n-1}}{8n}\epsilon, \text{for } k \neq j, 0 < x_k < 1/q\}$$

And since

$$[0, 1/q]^n \setminus (0, 1/q)^n \subseteq \bigcup_j A_j \cup \bigcup_k B_k$$

Thus

$$\begin{aligned} m^*([0,1/q]^n \setminus (0,1/q)^n) &\leq m^*(\bigcup_j A_j \cup \bigcup_k B_k) \\ &\leq \sum_{j=1}^{n} vol(A_j) + \sum_{k=1}^{n} vol(B_k) \\ &= 2\epsilon \frac{q^{n-1}}{8n} \frac{n}{q^{n-1}} + 2\epsilon \frac{q^{n-1}}{8n} \frac{n}{q^{n-1}} \\ &= \frac{\epsilon}{2} < \epsilon \end{aligned}$$

This is true for all $\epsilon > 0$, thus

$$m^*([0,1/q]^n \setminus (0,1/q)^n) = 0$$

Thus, both boxes are measure q^{-n}. □

Problem 5 (Exercise 7.4.1)

If A is an open interval in $\mathbf{R}$, show that $m^*(A) = m^*(A \cap (0, \infty)) + m^*(A \setminus (0, \infty))$.

Proof. Let $A = (a, b) \subset \mathbf{R}$ be an open interval. First, we know that the open box A has outer measure $m^*(A) = b - a$ We prove the result by dividing A into three cases.

1. when $0 \leq a < b$, $A \cap (0, \infty) = A$, $A \setminus (0, \infty) = \emptyset$, thus

$$\begin{aligned} m^*(A \cap (0, \infty)) + m^*(A \setminus (0, \infty)) &= m^*(A) + m^*(\emptyset) \\ &= m^*(A) \end{aligned}$$

2. when $a < 0 < b$, $A \cap (0, \infty) = (0, b)$, $A \setminus (0, \infty) = (a, 0]$. Since

$$(a, -\epsilon) \subset (a, 0) \subset (a, \epsilon)$$

for any $\epsilon > 0$, make ϵ smaller if necessary. Thus, by monotonicity,

$$\begin{aligned} &m^*((a, -\epsilon)) \leq m^*((a, 0)) \leq m^*((a, \epsilon)) \\ &\to -\epsilon - a \leq m^*(a, 0) \leq \epsilon - a \end{aligned}$$

Thus $m^*(a, 0) = -a$ and since $A \cap (0, \infty) = (0, b) = b$, thus,

$$\begin{aligned} m^*(A \cap (0, \infty)) + m^*(A \setminus (0, \infty)) &= b + (-a) \\ &= m^*(A) \end{aligned}$$

3. when $a < b \leq 0$, $A \cap (0, \infty) = \emptyset$, $A \setminus (0, \infty) = A$, thus

$$\begin{aligned} m^*(A \cap (0, \infty)) + m^*(A \setminus (0, \infty)) &= m^*(\emptyset) + m^*(A) \\ &= m^*(A) \end{aligned}$$

□

Problem 6 (Exercise 7.4.2)

If A is an open box in $\mathbf{R}^n$, and E is the half-plane $E := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_n > 0\}$, show that $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$.

Proof. Let A be an open box such that $A = \prod_{i=1}^n (a_i, b_i)$, let E be the half-plane. Then Since $A = (A \cap E) \cup (A \setminus E)$, thus by subadditivity of the outer measure

$$m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E)$$

Now we try to show that $m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E)$. First, note that A can be written as $A = A_1 \times A_2$, where $A_1 \subseteq \mathbf{R}^{n-1}$ and $A_2 \subseteq \mathbf{R}$. Similarly, E can also be written into $E = E_1 \times E_2$ where $E_1 \subseteq \mathbf{R}^n, E_2 \subseteq \mathbf{R}$. Then

$$\begin{aligned} A \cap E &\subseteq A_1 \times (A_2 \cap E_2) \\ \to m^*(A \cap E) &\leq m^*(A_1 \times (A_2 \cap E_2)) \\ &= m^*(A_1)m^*(A_2 \cap E_2) \end{aligned}$$

similarly,

$$\begin{aligned} A \setminus E &\subseteq A_1 \times (A_2 \setminus E_2) \\ \to m^*(A \setminus E) &\leq m^*(A_1 \times (A_2 \setminus E_2)) \\ &= m^*(A_1)m^*(A_2 \setminus E_2) \end{aligned}$$

Thus combining the two arguments

$$\begin{aligned} m^*(A \cup E) + m^*(A \setminus E) &\leq m^*(A_1)m^*(A_2 \cap E_2) + m^*(A_1)m^*(A_2 \setminus E_2) \\ &= m^*(A_1)(m^*(A_2 \cap E_2) + m^*(A_2 \setminus E_2)) \\ &= m^*(A_1)m^*(A_2) \\ &= m^*(A) \end{aligned}$$

Hence

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

□

Problem 7 (Exercise 7.4.3)

Prove Lemma 7.4.2. (Half-spaces are measurable). The half-space

$$\{(x_1, \dots, x_n) \in \mathbf{R}^n : x_n > 0\}$$

is measurable.

Proof. Let $A \subseteq \mathbf{R}$, E denote the half-space. Then, note that since $A = (A \cap E) \cup (A \setminus E)$, thus

$$m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E)$$

Now, we try to show $m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E)$. Let $\epsilon > 0$, there exist open cover of A, (A_j) such that $\sum_j vol(A_j) \leq m^*(A) + \epsilon$ And we know that

$$\begin{aligned} A \cap E &\leq E \cap \bigcup_j A_j \\ \to m^*(A \cap E) &\leq m^*(E \cap \bigcup_j A_j) \\ &= m^*(\bigcup_j A_j \cap E) \\ &\leq \sum_j vol(A)_j \cap E) \end{aligned}$$

$$\begin{aligned} A \setminus E &\subseteq (\bigcup_j A_j) \setminus E \\ \to m^*(A \setminus E) &\leq m^*((\bigcup_j A_j) \setminus E) \\ &= m^*(\bigcup_j A_j \setminus E) \\ &\leq \sum_j vol(A_j \setminus E) \end{aligned}$$

combining the two arguments we get

$$\begin{aligned} m^*(A \cap E) + m^*(A \setminus E) &\leq \sum_j m^*(A_j \cap E) + \sum_j m^*(A_j \setminus E) \\ &= \sum m^*(A_j) \leq m^*(A) + \epsilon \end{aligned}$$

This is true for all $\epsilon > 0$, thus $m^*(A \cap E) + m^*(A \setminus E) \leq m^*(A)$. Hence

$$m^(A) = m^*(A \cap E) + m^*(A \setminus E)$$

So the half-space is measurable. ☐

Problem 8 (Exercise 7.4.4)

Prove Lemma 7.4.4

(a)

If E is measurable, then $\mathbf{R}^n \setminus E$ is also measurable.

Proof. We want to show that for any set $A \subset \mathbf{R}^n$

$$m^*(A) = m^*(A \cap (\mathbf{R}^n \setminus E)) + m^*(A \setminus (\mathbf{R}^n \setminus E))$$

Let $A \subset \mathbf{R}^n$, by bayes diagram, we get

$$\begin{aligned}
&m^*(A \cap (\mathbf{R}^n \setminus E)) + m^*(A \setminus (\mathbf{R}^n \setminus E)) \\
=&m^*(A \setminus E) + m^*(A \cap E) = m^*(A)
\end{aligned}$$

Thus $\mathbf{R}^n \setminus E$ is measurable. □

(b)

(Translation invariance) If E is measureable, and $x \in \mathbf{R}^n$, then $x + E$ is also measurable, and $m(x + E) = m(E)$.

Proof. We want to show for any set $A \subset \mathbf{R}^n$, we have

$$m^*(A) = m^*(A \cap (E + x)) + m^*(A \setminus (E + x))$$

Let $A \subset \mathbf{R}^n$, by translation invariance property of the outer measure,

$$\begin{aligned}
m^*(A) = m^*(A - x) &= m^*((A - x) \cap E) + m^*((A - x) \setminus E) \\
&= m^*((A - x) \cap E + x) + m^*((A - x) \setminus E + x) \\
&= m^*(A \cap (E + x)) + m^*(A \setminus (E + x))
\end{aligned}$$

Thus $E + x$ is measurable and the measure would be $m^*(E + x)$, hence by translation invariance of outer measure,

$$m^*(E + x) = m^*(E)$$

□

(c)

If E_1 and E_2 are measurable, then $E_1 \cap E_2$ and $E_1 \cup E_2$ are measurable.

Proof. Let $A \subset \mathbf{R}^n$, we first divide A into coordinates shown in the following table

	E_1^c	E_1
E_2	A_{-+}	A_{++}
E_2^c	A_{--}	A_{+-}

Table 1: where
$A_{++} = A \cap E_1 \cap E_2$,
$A_{+-} = A \cap E_1 \cap E_2^c$,
$A_{-+} = A \cap E_1^c \cap E_2$,
$A_{--} = A \cap E_1^c \cap E_2^c$

We first prove the case that $E_1 \cap E_2$ is measurable. That is, WTS that for any $A \subset \mathbf{R}^n$

$$m^*(A) = m^*(A \cap (E_1 \cap E_2)) + m^*(A \cap (E_1 \setminus E_2))$$

which is equivalent to showing that

$$m^*(A) = m^*(A_{++}) + m^*(A_{-+} \cup A_{--} \cup A_{+-})$$

Since E_1 is measurable, we know that

$$\begin{aligned}
m^*(A) &= m^*(A \cap E_1) + m^*(A \setminus E_1) \\
&= m^*(A_{++} \cup A_{+-}) + m^*(A_{-+} + A_{--}) \\
&= m^*((A_{++} \cup A_{+-}) \cap E_2) + m^*((A_{++} \cup A_{+-}) \setminus E_2) \\
&\quad + m^*((A_{-+} + A_{--}) \cap E_2) + m^*((A_{-+} + A_{--}) \setminus E_2) \\
&= m^*(A_{++}) + m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})
\end{aligned}$$

Thus if we show that $m^*(A_{-+} \cup A_{--} \cup A_{+-}) = m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})$ then we're done. But then using that E_1 and E_2 are measurable, we know

$$\begin{aligned}
&m^*(A_{-+} \cup A_{--} \cup A_{+-}) \\
=&m^*((A_{-+} \cup A_{--} \cup A_{+-}) \cap E_1) + m^*((A_{-+} \cup A_{--} \cup A_{+-}) \setminus E_1) \\
=&m^*(A_{+-}) + m^*(A_{--} \cup A_{-+}) \\
=&m^*(A_{+-}) + m^*((A_{--} \cup A_{-+}) \cap E_2) + m^*((A_{--} \cup A_{-+}) \setminus E_2) \\
=&m^*(A_{+-}) + m^*(A_{-+}) + m^(A_{--})
\end{aligned}$$

Thus

$$m^*(A) = m^*(A \cap (E_1 \cap E_2)) + m^*(A \cap (E_1 \setminus E_2))$$

$E_1 \cap E_2$ is measurable.

Now to prove that $E_1 \cup E_2$ is measurable, we try to show that for every $A \subset \mathbf{R}^n$

$$m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \setminus (E_1 \cup E_2))$$

which is equivalent to showing

$$m^*(A) = m^*(A_{++} \cup A_{+-} \cup A_{-+}) + m^*(A_{--})$$

Thus if we show that

$$m^*(A_{++} \cup A_{+-} \cup A_{-+}) = m^*(A_{++}) + m^*(A_{+-}) + m^*(A_{-+})$$

then we are done. By similar argument, since E_1, E_2 are both measurable, thus

$$m^*(A_{++} \cup A_{+-} \cup A_{-+}) = m^*(A_{++}) + m^*(A_{+-}) + m^*(A_{-+})$$

And so $E_1 \cup E_2$ is measurable. □

(d)

(Boolean algebra property) If $E_1, E_2, \ldots, E_N$ are measurable, then $\bigcup_{j=1}^N E_j$ and $\bigcap_{j=1}^N E_j$ are measurable.

Proof. We prove the claim by combining mathematical induction with property (c) of measurable sets.

For $N = 2$, property (c) shows that both $E_1 \cap E_2$ and $E_1 \cup E_2$ are measurable. Suppose when $N = n$, that $\bigcup_{j=1}^N E_j$ and $\bigcap_{j=1}^N E_j$ are both measurable, then

$$\bigcup_{j=1}^{N+1} E_j = \bigcup_{j=1}^{N} E_j \cup E_{N+1}$$

and

$$\bigcap_{j=1}^{N+1} E_j = \bigcap_{j=1}^{N} E_j \cap E_{N+1}$$

, since $\bigcup_{j=1}^N E_j, \bigcap_{j=1}^N E_j, E_{N+1}$ are all measurable, applying (c), we get that $\bigcap_{j=1}^{N+1} E_j$ and $\bigcup_{j=1}^{N+1} E_j$ are both measurable.

Thus by mathematical induction, for all $N \in \mathbf{N}_{\geq 2}$, $\bigcup_{j=1}^N E_j$ and $\bigcap_{j=1}^N E_j$ are measurable. □

(e)

Every open and closed boxes are measurable.

Proof. Since boxes are translations and intersection of half spaces, thus measurable. □

(f)

Any set E of outer measure zero (i.e. $m^*(E) = 0$) is measurable.

Proof. By finite subadditivity, let $A \subset \mathbf{R}^n$,

$$m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E)$$

thus it suffices to show

$$m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E)$$

By monotonicity, since $A \cap E \subseteq E$, thus

$$m^*(A \cap E) \leq m^*(E) = 0$$

, thus

$$m^*(A \cap E) = 0$$

. Thus now if we show that $m^*(A) \geq m^*(A \setminus E)$, then we're done, but this is true also by monotonicity, since $A \setminus E \subseteq A$. Thus

$$m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E)$$

and so E is measurable. $\square$