Essay

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In this essay, I will talk about some examples in the book "Counterexamples in Analysis", specifically, some examples from the chapter "Sets and Measure on the Real Axis". I will introduce three examples I find very interesting and relates to the measure content in class. Most examples are related to Cantor sets, which yields interesting properties that defies intuition.

The examples would be mainly about dense sets, perfect sets. Before going to the examples, we first define what it meant by a set to be dense and perfect.

Definition (Isolated points). a point x is called an isolated point of a subset S in a topological space X if x is an element of S and there exists a neighborhood of x which does not contain any other points of S.

Example. Here's an example of an isolated point, in $\mathbf{R}$, let $S \subset \mathbf{R}$, $S = \{0\} \cup [1, 2]$, then 0 is an isolated point of S.

Definition (Limit point). a point x is called a limit point of a subset S in a topological space X if x is an element of S and for every neighborhood of x, the neigborhood contains at least one point in S different from x itself.

Remark. A isolated point of a set is not a limit point of a set.

Definition (Perfect set). A set is perfect if it is closed and has no isolated points. Or in other words, every element in the perfect set is a limit point.

Example. Here we list out some perfect sets on the real line

- Empty set is a perfect set: True, since empty set is closed, and no element is present in the set making the statement vacuously true.

- Closed intervals are perfect sets: True since closed, and for every radius, the open ball around every point contains an element of the closed interval.

- The cantor set is perfect: True since the cantor set is union

Definition (Closure). The closure of S is the smallest closed set containing S

Definition (Interior). A point x in the set S is an interior point of S if there exist an open ball $B_r(x)$ for some $r > 0$ such that $B_r(x) \subset S$.

Definition (Nowhere dense set). A subset of a topological space is called nowhere dense or rare if its closure has empty interior.

Example. Here we list some examples of a nowhere dense set.

- $\{1/n | n \in \mathbf{N}\}$ is nowhere dense in $\mathbf{R}$.

- $\mathbf{Z}$ is nowhere dense in $\mathbf{R}$.

1. A perfect nowhere dense set

It is straightforward to have a perfect that is not nowhere dense, such examples is like a closed interval, since a closed interval is perfect but the interior is not empty. Or to have a nowhere dense set that is not perfect, such as the natural numbers $\mathbf{Z}$ in $\mathbf{R}$, it is not perfect for $\mathbf{Z}$ has isolated points in $\mathbf{R}$. But can we find a set that is both nowhere dense and perfect?

It turns out that the Cantor set is nowhere dense and perfect. Recall that the cantor set $\mathcal{C}$ is created by iteratively deleting the open middle third from a set of line segments. $\mathcal{C}$ is perfect since it is closed (complement of countable union of open sets), and there is no isolated points. The cantor set is nowhere dense, since suppose it is not nowhere dense, there is an interval $(a, b) \subset \mathcal{C}$. But then $\mathcal{C}$ has zero measure, a contradiction, so cantor set is nowhere dense.

2. An uncountable set of measure zero

It is also intuitive to come up with uncountable sets with positive measure, such as the interval $m(0, 1) = 1$, or a countable set with measure zero, such as the set of natural numbers $\mathbf{N}$. But can we find an uncountable set of measure zero?

The Cantor set $\mathcal{C}$ is way to go. It is measure zero since the the measure of the cantor set is upper bounded by the sequence $a_n = \left(\frac{2}{3}\right)^n$ which by repeating the process infinite number of times converges to zero. Thus $m(\mathcal{C}) \leq \lim a_n = 0$. To show that $\mathcal{C}$ is uncountable. First one show that

$$\mathcal{C} = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} \,\middle|\, a_n \in \{0, 2\} \right\}$$

Then we can think of $\mathcal{C}$ as a tenary (base three) base expansion of all of the numbers between [0,1], the number of tenary representations are uncountable, thus the cantor set $\mathcal{C}$ is uncountable.

3. A set of measure zero whose difference set contains a neighborhood of the origin

First we define the difference set.

Definition (difference set)**.** The difference set $D(A)$ of a nonempty set A is the set of all differences between members of A.

$$D(A) = \{x - y | x \in A, y \in A\}$$

One of the statements states that

Theorem 1. *Let $A \subset \mathbf{R}^n$ be a Lebesgue-measurable set of positive measure. Then we follow that $D(A) = \{x - y | x \in A, y \in A\}$ is a neighborhood of zero, i.e. contains an open ball around zero.*

Example. $S = [1, 2]$, then 0 is an interior point of $D(S)$.

We'd like to know whether we could say the same thing for measure zero sets, and intuitively, we can't. $\mathbf{Z}$ is measure zero, but 0 is not an interior point. Now we want to find if there's a set that satisfies the conclusion of the theorem even when the set has zero measure. It turns out that the cantor set is measure zero, but the difference contains a neighborhood of the origin. We make the following proposition.

Proposition 1. *The difference set $D(\mathcal{C})$ of a cantor set $\mathcal{C}$ is the closed interval $[-1, 1]$. That is*
$$D(\mathcal{C}) = [-1, 1]$$

Here we demonstrate two ways of proving the claim. One according to the fact that the any element in Cantor set can be written as ternary expansion that contains only digits 0 and 2. The outline of the proof can be described in two steps

1. First, we'll see that any number with a finite ternary expansion can be written as the difference of two elements from the Cantor set.

2. Next, we'll use compactness of the $\mathbf{C}$ to cover numbers in [-1,1] with infinite ternary expansion.

Proof.

Proposition 2. *Any number with a finite ternary expansion can be written as the difference of two elements from the Cantor set*

Proof for proposition 2:

Let A be a real number from $[0, 1]$ with an infinite ternary expansion. $A = 0.a_1a_2a_3\ldots$ is a limit of its partial expansions $A_k = 0.a_1a_2\ldots a_k$. Using proposition 2, there exist elements of the $\mathcal{C}$, B_k and C_k such that $A_k = B_k - C_k$. Since the Cantor set is compact (closed and bounded in $\mathbf{R}$). Consider the sequence $\{B_i\}$. It contains a convergent sequence $\{B_{ij}\}$. The sequence $\{C_{ij}\}$, in turn, contains a convergent subsequence $\{C_{ijr}\}$. Since any subsequence of a convergent sequence is also convergent (and to the same limit), the subsequence $\{B_{ijr}\}$ is also convergent. Denote their limits B and respectively C. B and C are both elements of the Cantor set and, additionally, $A = B - C$, which completes the proof. $\square$

Another proof follows the argument of Counterexamples in Analysis. The key idea is to show that every line $y = x + a$, where $a \in [-1, 1]$, intersects the product $\mathcal{CC}$. If (c, b) is the point of intersection, we automatically have $a = b - c$, where both $b, c \in \mathcal{C}$ belong to the Cantor set. Graphically speaking, figure 1
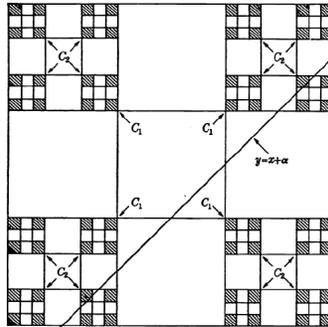

Figure 1: The graph of the proof

Proof. $y = x + a$ intersects at least one of the 4 squares with side $1/3$ in the corner. Denote any of these as S_1. Similarly, since the slope of the line is 1, the line intersects with one of the corners of S_1, denote the corner S_2. Since the sequence of squares are compact, and $S_1 \supset S_2 \supset \ldots$, the line with limited domain is also compact in $\mathbf{R}^2$. Then we have (line $\cap \bigcap S_n \neq \emptyset$) by the properties of countable intersections of compact sets. $\square$

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