Math 105 Lec Feb. 15

- Simple functions

  - $E \subset \mathbb{R}^n$ measurable
  - $\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & \text{else} \end{cases}$ indicator function "characteristic function"

  - simple func. are finite linear combinations of indicators

$$f(x) = \sum_{i=1}^{N} c_i \mathbb{1}_{E_i}(x) \qquad E_i \subset \mathbb{R}^n, \text{ meas.}$$

$$\int \mathbb{1}_E = m(E)$$

$$\int \sum c_i \mathbb{1}_E = \sum c_i m(E_i)$$

  - Finally, for measurable function (nonegative)

    ① find simple func. $f_n : \mathbb{R}^n \to [0,\infty)$

    s.t. $f_n \nearrow f$ pointwise

    ② define $\int f = \lim_{n\to\infty} \int f_n$

Tau

7.5 measurable function: $f : \mathbb{R}^n \to \mathbb{R}$ is mearurable

if $\forall \; V \subset \mathbb{R}$, $f^{-1}(V)$ is measurable

$\uparrow$
open.

- more generally, $\Omega \subset \mathbb{R}^n$ a measurable set

$f: \Omega \to \mathbb{R}$, then f is measurable
if $f^{-1}(\text{open})$ is measurable

<u>Lemma</u>: $f: \Omega \to \mathbb{R}^n$ meas. $f: \Omega \to \mathbb{R}$ continuous
then f is meas.

<u>pf</u>: $\forall V \subset \mathbb{R}$ open, we have $f^{-1}(V) \subset \Omega$ is open in Ω

i.e. $f^{-1}(V) = W \cap \Omega$ where $W \subset \mathbb{R}^n$ is open
but W and Ω are measurable, $\therefore W \cap \Omega$ is meas.

<u>Lemma</u>: if $\underset{\text{meas.}}{f: \Omega \to \mathbb{R}^k}$ and $\underset{\text{cont.}}{g: f(\Omega) \to \mathbb{R}^L}$, then
$g \circ f$ is measurable

pf:
$$\Omega \xrightarrow{f} f(\Omega) \xrightarrow{g} \mathbb{R}^L$$
$$\underset{\text{meas.}}{f^{-1}(g^{-1}(V))} \longrightarrow \underset{\text{open}}{g^{-1}(V)} \longrightarrow \underset{\text{open}}{V}$$
$\because g$ is continuous.

Cor: if f is meas. $\mathbb{R} \to \mathbb{R}$, then $|f|$ is meas.

$$\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{|\cdot|} \mathbb{R}$$

(2) if $f_1\ f_2$ are meas. $\mathbb{R} \to \mathbb{R}$
then $f_1 + f_2$ is measurable

Lemma: a function $f: \mathbb{R}^k \to \mathbb{R}^n$ $f = (f_1, \dots, f_n)$

is meas. iff $f_1, \dots, f_n$ are all meas.

pf: $\Rightarrow$ $\because f_i = \pi_i \circ f$, $\pi_i : \mathbb{R}^n \to \mathbb{R}$

$\therefore f_i$ is meas. $\forall i$ proj to ith component

$\Leftarrow$ to prove f is measurable, suffice to check f^{-1}(open boxes are measurable)

$\because$ any open set is a countable union of open boxes.

$$f^{-1}\big((a_1,b_1)\times(a_2,b_2)\times\cdots\times(a_n,b_n)\big)$$
$$= \bigcap_{i=1}^{n} f_i^{-1}\big((a_i,b_i)\big)$$
$$= \bigcap_{i=1}^{n} (\text{measuable sets}) = \text{meas.}$$

Let $f: \mathbb{R} \to [0,\infty)$

$u f$ measurable $\iff$ $\forall$ V open in $\mathbb{R}^+$

$f^{-1}(V)$ is measurable

$\Updownarrow$

$\forall (a,\infty)$ $a \geq 0$

$f^{-1}\big((a,\infty)\big)$ is measurable

- $f^{-1}\big((-\infty,a]\big)$
  $= \big[f^{-1}((a,\infty))\big]^c$ is measurable

- $a<b$ then $f^{-1}\big((a,b]\big) = f^{-1}\big((a,\infty)\setminus(b,\infty)\big) = f^{-1}((a,\infty)) \setminus f^{-1}((b,\infty))$

☆ warning : (1) if $E \subset \mathbb{R}^2$ measurable it doesn't mean $\pi_i(E)$ is measurable

(2) if $f: \mathbb{R} \to \mathbb{R}^2$ continuous func., $E \subset \mathbb{R}^2$ measurable

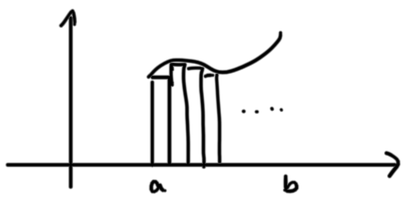
$f^{-1}(E) \subset \mathbb{R}$ may not be measurable.

e.g. $f \begin{cases} 1 & x \in A \\ 0 & \text{o.w.} \end{cases}$ graph of f is measurable

$\therefore$ preimage of measurable sets are not necessarily measurable for measurable functions

Riemann Integral:

$f: [a,b] \to [0,\infty)$ continuous



- what is cantor set?
- what is homomorphism.
- what is Leitchz

- Lebesgue Integral
  - $E \subset \mathbb{R}$ be a measurable set

    $\mathbb{1}_E(x)$ 

  - Def: if $f: \mathbb{R} \to \mathbb{R}$ is a measurable func. and $f(\mathbb{R})$ is a finite set, then f is called a simple function

  - Lemma if f is a simple func., then.

    $\exists$ $E_1, \dots, E_n$ disjoint meas. subsets of $\mathbb{R}$

    $c_1, \dots, c_N \in \mathbb{R}$

    s.t. $f(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(x)$

pf: Let $f(\mathbb{R}) = \{c_1, c_2, \dots, c_n\}$

$\forall c_i$ choose ε_i small enough such that $(c_i - \varepsilon_i, c_i + \varepsilon_i) \cap f(\mathbb{R}) = \{c_i\}$

then $f^{-1}\big((c_i - \varepsilon_i, c_i + \varepsilon_i)\big) := E_i$ meas. Thus $f(x) = \sum c_i \mathbb{1}_{E}(x)$

• prop : the set of simple functions form a vector space i.e.

- $\forall c \in \mathbb{R}$ $\forall f$ simple func. $c \cdot f$ is simple
- $\forall f, g$ simple func. $f + g$ is simple

• Def : Let f be a simple func. $\int f = \sum_{\substack{\lambda \in f(\mathbb{R}) \\ \lambda > 0}} \overset{\text{height.}}{\lambda} \cdot \underbrace{m(f^{-1}(\lambda))}_{\text{base width.}}$

$f : \mathbb{R} \to [0, \infty)$

• Lemma: ① $\forall c > 0$, $f : \mathbb{R} \to [0, \infty)$ simple, $\int cf = c \int f$

② if $f, g : \mathbb{R} \to [0, \infty)$ simple. $\int f + \int g = \int f + g$.

Pf ② : suffices to prove that $\forall E_i$ meas. (may not be disjoint)

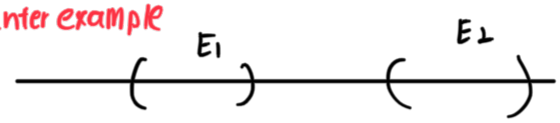
✓ $\int \sum c_i \mathbb{1}_{E_i}(x) = \sum c_i m(E_i)$

in tao

~~Let $E_{ij} = E_i \cap E_j$, $c_{ij} = c_i + c_j$~~

$\sum c_i \mathbb{1}_{E_i}(x) = \sum_{1 \le i < j \le n} c_{ij} \mathbb{1}_{E_{ij}}(x)$ → not true

Counter example



Prop : Let $f : \mathbb{R} \to [0, \infty)$ be measurable

then $\exists f_n$ seq. of nonneg. simple functions of bounded support (places where function is nonzero)

s.t. $f_n \nearrow f$ pointwise

pf:

$$f_n(x) = \max\left\{ \frac{j}{2^n} \,\middle|\, \frac{j}{2^n} \le \min\{f(x), 2^n\},\ j \in \mathbb{Z} \right\}$$

$$f_n(x) = \frac{j}{2^n}$$

$$\Leftrightarrow \frac{j}{2^n} \le f(x) < \frac{j+1}{2^n}$$

$$\Leftrightarrow x \in f^{-1}\left(\left[\frac{j}{2^n}, \frac{j+1}{2}\right)\right)$$

$$f_n(x) = 2^n \Leftrightarrow f(x) \ge 2^n$$

$$\Leftrightarrow x \in f^{-1}([2^n, \infty))$$

prop: $f: \mathbb{R} \to [0,\infty)$ is measurable

$\Leftrightarrow$ f is a pointwise limit of a sequence of simple functions

pf: $\Rightarrow$ by above construction

$\Leftarrow$ we prove that if f_n is a sequence of non-measurable functions $f_n \to f$ pointwise. f is meas.

need to check $f^{-1}([a,\infty)) = \bigcap_{n=1}^{\infty} \underline{f_n}^{-1}([a,\infty))$ meas.

$\underline{f_n} \nearrow f_n$