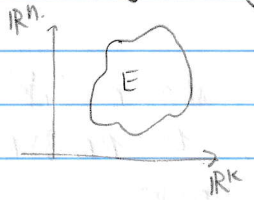
Lec.

today : • finish up slice theorem
• Lebegue integral.

Recall :



$E \subset \mathbb{R}^k \times \mathbb{R}^n$

$x \in \mathbb{R}^k \qquad E_x = E \cap \{x\} \times \mathbb{R}^n \subset \mathbb{R}^n$

• Thm : $m(E) = 0$ iff almost every slice has measure 0

i.e. if we define $Z = \{x \mid m(E_x) > 0\}$.

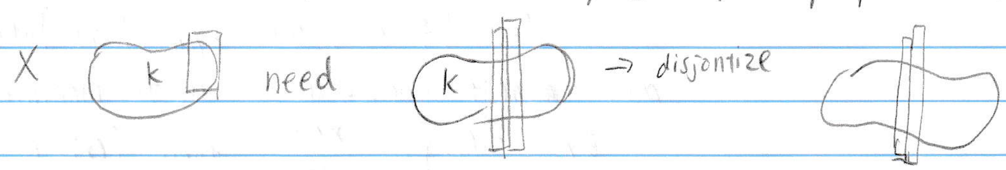
then $m(E) = 0 \quad Z \in \mathbb{R}^k$

a.

last time $\Leftarrow$ ① may assume $Z = \phi$, E is bounded

② use inner approximation of E by compact subset K.

③ try to cover K by finitely many open boxes.

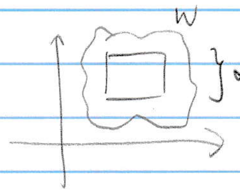


vertically long enough) then disjontize.

④ $m(K) \le \sum m(U_i) \times m(V_i) < \varepsilon$

$\sum m(U_i) < 1 < \varepsilon$

⑤ $m(E) < m(K) + \varepsilon < 2\varepsilon$



b. $\Rightarrow$ $\forall W$ open, $\overset{\star\ \text{bounded}\ \forall \alpha > 0}{\wedge}$ at X $\qquad W \subset \mathbb{R}^k \times \mathbb{R}^n$.

$X_\alpha = \{x \in \mathbb{R}^k \mid m(W_x) > \alpha\} \qquad m(W) \ge m(\square)$

$\underset{n+k}{m}(W) \ge m_k(X_\alpha) \cdot \alpha \qquad = m(X_\alpha) \cdot \alpha$.

pf.

① $\forall x \in X_\alpha$, get compact set $K_x \subset W_x$. $\quad m(K_x) > \alpha \quad \begin{matrix} m(W_{x'}) \ge m(K_x) \\ > \alpha. \end{matrix}$

$\{x\} \times K_x \subset \overset{\mathbb{R}^k}{U(x)} \times \overset{\mathbb{R}^n}{V(x)} \subset W$

$\llcorner$ do union of compact boxes $\Rightarrow X_\alpha$ is open

$\because U(x) \subset X_\alpha$ indeed $\quad W_{x'} \supset V(x) \supset K_x$.

compact subset

② $\forall K' \subset\subset X_\alpha$ cpt

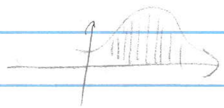
$\because X_\alpha \subset \bigcup_{x \in X_\alpha} u(x)$

$\therefore$ we have finite subcover for K'

$u_1 = u(x_1) \quad u_2 = u(x_2) \setminus u(x_1), \ldots$

u_i disjoint

$m(W) \geq m\left(\bigsqcup_{i=1}^{N} u_i \times V(y_i) \right) \geq \sum m(u_i) \times \alpha \geq m(K')$



Pugh 6.6

Let $f : \mathbb{R} \to [0,\infty)$.

Def: undergraph $u(f) = \{ (x,y) \mid 0 \leq y < f(x) \}$

- we say f is measurable if $u(f)$ is a measurable subset
- $\int f := m(u(f))$ (possibley $+\infty$)

if $\int f < \infty$, we say f is Integrable.

a.e. almost everywhere = "up to measure zero set"

Ex: $f(x) \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{else.} \end{cases}$ then $f(x) = 0$ almost everywhere.

- Thm. Let $f_n : \mathbb{R} \to [0,\infty)$ be a sequence of measurable fun. and $f_n \nearrow f$ a.e. as $n \to \infty$, Then $\int f_n \nearrow \int f$

there is a null set $Z \subset \mathbb{R}$ s.t. $x \in \mathbb{R} \setminus Z$

$\lim_{n \to \infty} f_n(x) = f(x) \quad f_{n+1}(x) \geq f_n(x)$

- pf: $f_n \nearrow f \Rightarrow u_n(f_n) \nearrow u(f)$

$A = \bigcup A_n$.

$A_n \nearrow A$ means $A_n \subset A_{n+1} \subset \cdots$

- Def completed undergraph $\hat{u}(f) = \{ (x,y) \mid 0 \leq y \leq f(x) \}$

prop: $u(f)$ is measurable iff $\hat{u}f$ is measurable.

$m(u(f)) = m(\hat{u}(f))$

Fact (Pugh 6.3)

if $T: \mathbb{R}^n \to \mathbb{R}^n$ affine linear transformation

$T(x) = Ax + b$ $\quad A = \square \quad x = () \quad b = ()$

then $A \subset \mathbb{R}^n$ is measurable

then $m(T(E)) = |T| \cdot m(E)$

$|T| = |\det(A)|$

pf: suppose $u f$ is measurable, $\forall n > 0$ integer

$$u(f) \subset \hat{u}(f) \subset \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 + 1/n \end{pmatrix} u(f)}_{R_n(f)} + \mathbb{R} \times \{0\}$$

(stretch graph a little will contain boundary)

$\bigcap_{n=1}^{\infty} R_n(f)$ is measurable

$$m(\cap R_n(f)) = \lim m(R_n(f))$$
$$= \lim (1 + 1/n)\, m(u(f)) = m(u(f)) = m(\hat{u}(f))$$

proposition

If $f_n : \mathbb{R} \to [0, \infty)$ is a seq of integrable fun. that $f_n \searrow f$ a.e

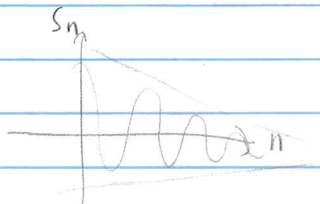
$\int f =$ then $\int f_n \searrow \int f$

pf: $m(u(f)) = m(\hat{u}(f)) \swarrow m(\hat{u}(f_n)) = m(u(f_n)) = \int f_n$

Recall if a_n is a bounded sequence.

$\bar{a}_n := \sup\{a_m : m \geq n\}$

$\underline{a_n} := \inf\{m \geq n\}$



Now if $f_n(x)$ is a sequence of functions

$\bar{f}_n(x) = \sup\{f_m(x), m \geq n\}$

- Prop. $u(\bar{f}_n) = \bigcup_{k \geq n} u(f_k)$

  $\hat{u}(f_n) = \bigcap_{k \geq n} \hat{u}(f_k)$

pf: exercise.

· Thm. · suppose we have a sequence of f_n, measurable.

$f_n \to f$ a.e.

· $\exists\, g: \mathbb{R} \to [0,\infty)$ s.t. $g(x) \geq f_n(x)$ a.e. $\int g < \infty$

then $\int f_n \to \int f$. $\because u(f_n) \subset u(g)$

$\therefore m(u(f_n)) \leq m(u(g)) < \infty$

pf. · $u(\underline{f_n}) \subset u(f_n) \subset \hat{u}(\overline{f_n})$

· $\underline{f_n} \nearrow f$ $\quad \overline{f_n} \searrow f$

$\int \underline{f_n} = \int f$, $\lim \int \underline{f_n} = \lim \int \overline{f_n} = \int f$.

example: (need for g)

$f_n = n \cdot \mathbb{1}_{[0,1/n]}(x)$.

$\lim f_n(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$

thus $f_n \to 0$

$\int f_n = 1$ $\int f = 0$