Lec.

Last time:

- Lebesgue criteria for measurability $(\cap \text{ open})$

  $E \subset \mathbb{R}^n$ is measurable if $\exists$ a G_δ set G

  F_σ set F s.t. $G \supset E \supset F$, $m(G \setminus F) = 0$

  $(\cup \text{ closed})$

We call G is an "hull" of F

F is a "kernel" of F these are unique up to a null set

(i.e. G_1, G_2 are nulls of F, then $G_1 \setminus G_2$, $G_2 \setminus G_1$ are null)

- Lemma: ① If $G_1, G_2, \dots$ are G_δ sets

  then: $\cap G_i$ is a G_δ set

  $G_1 = \bigcap_{j=1}^{\infty} U_{1j}$ $\quad G_2 = \bigcap_{j=1}^{\infty} U_{2j}$

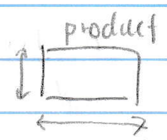
  $G_1 \cap G_2 = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} U_{ij}$

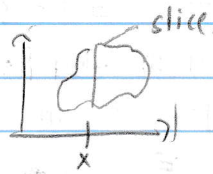
  still countably many open $\therefore$ G set

  (union cannot preserve)

  ② similarly $F_1, F_2, \dots$ are F_σ sets

  the $\bigcup_{j=1}^{\infty} \bigcup_{i=1}^{\infty} F_{ij}$ is F_σ.





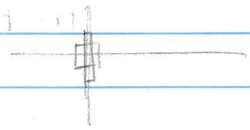
Product & slices:

Thm: if $E \subset \mathbb{R}^n$, $F \subset \mathbb{R}^k$ are measurable sets,

then $m(E \times F) = m(E)\, m(F)$

if $m(E) = 0$ or $m(F) = 0$ then $m(E \times F) = 0$

hyperplane $m(\{a\} \times \mathbb{R}) = 0$ $n=1, k=1$



• Lemma if $E \subset \mathbb{R}$ has $m(E)$, then $m(E \times \mathbb{R}) = 0$

pf: $\forall n > 0$ integer, we will cover E by collection of boxes with total area $\frac{\varepsilon}{2^{2n}}$ $\{B_{ni}\}_{i=1}^{\infty}$

$\bigcup_{i=1}^{\infty} B_{n,i} \supset E$ $\sum_{i=1}^{\infty} |B_{n,i}| \le \varepsilon/2^{2n}$

$\widetilde{B_{n,i}} = B_{n,i} \times (-2^n, 2^n)$ $\sum |\widetilde{B}_{n,i}| \le \varepsilon/2^n$

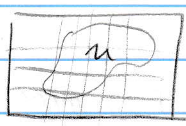
$\sum_i \sum_n |\widetilde{B}_{n,i}| \le \varepsilon.$

• Lemma. $m(E \times F) = m(E)\, m(F)$ for

- E, F are open boxes ? whendid we prove this?
- E, F are boxes
- E, F are opensets

• Lemma any open set $\mathcal{U} \subseteq \mathbb{R}^n$ can be written as a countable union of open boxes and a measure zero set.

pf: (take $n=2$)



consider unit open squares $(a, a+1) \times (b, b+1)$ $a, b \in \mathbb{Z}$.

for size 1 open boxes contained in $\mathcal{U}$, we take them.

for size __ that $\cap\, \mathcal{U} = \phi$, ignore them.

for any open box such that $B \cap \mathcal{U} \neq \phi$ we further divide B into 4 size $1/2$ pieces.

$E = \bigsqcup_{i=1}^{\infty} B_i + Z$

$F = \bigsqcup_{i=1}^{\infty} B_i' + Z'$

then $m^*(E \times F) = m\left(\left(\bigcup_{i=1}^{\infty} B_i \times \bigcup_{j=1}^{\infty} B_j'\right)\right)$

$= \sum_{i,j} m(B_i) \times m(B_j') = \left(\sum m(B_i)\right) \times \left(\sum_j m(B')\right)$

For E, F measurable subset of $\mathbb{R}$, now prove
$m(E \times F) = m(E) \times m(F)$

pf: assume E, F are bounded, $E \subset R_1$, $F \subset R_2$, R_i are open boxes.

• let $H_E \supset E \supset K_E$, $H_F \supset F \supset K_F$

null kernel

$m(H_E \setminus K_E) = 0 \quad m(H_F \setminus K_F) = 0$

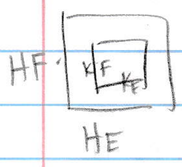
since H_E and H_F are G_δ sets
$m(H_E \times H_F) = m(H_E) \times m(H_F)$

claim $m(H_E \times H_F \setminus K_E \times K_F) = 0$ ✓

• $H_E \times H_F$ is G_δ, $K_E \times K_F$ is F_σ

$H_E = \cap u_n$

$H_F = \cap v_n \qquad H_E \times H_F = \bigcap_i \bigcap_j u_i \times v_j$



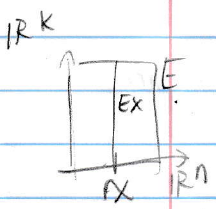
$H_E \times H_F \setminus K_E \times K_F$

$\subset (H_E \setminus K_E) \times H_F \cup H_E \times (H_F \setminus K_F)$

measure 0

$\therefore E, F$ is measurable.

$m(E \times F) = m(H_E \times H_F) = m(H_E) \times m(H_F) = m(E)\, m(F)$



Let $E \subset \mathbb{R}^n \times \mathbb{R}^k$ for any $x \in \mathbb{R}^n$ let $E_x = E \cap \{x\} \times \mathbb{R}^k \leq \{x\} \times \mathbb{R}^k \approx \mathbb{R}^k$

Thm: let $Z = \{x \in \mathbb{R}^n \mid m_{\mathbb{R}^k}(E_x) \neq 0\}$ is measure zero in $\mathbb{R}^n$.
then $m(E) = 0$

pf: let $(\tilde{E}) = E \setminus Z \times \mathbb{R}^k$, then $\because m(Z \times \mathbb{R}^k) = 0$

$\therefore m(\tilde{E}) = m(E)$

Assume E is bounded subset in $\mathbb{R}^2$. $(n=1, k=1)$

- Assume $E \subset [0,1]^2$ $m(E_x)=0$ $\forall x$
- $\forall \varepsilon > 0$, we want to show $m(E) < \varepsilon$

① find $K \subset E$, closed, s.t. $m(E \setminus K) \leq \varepsilon/2$

then K is closed & bounded $\Rightarrow K$ is cpt & $m(K_y)=0$ $\forall x$

② cover K by boxes of total area $\leq \varepsilon/2 \Rightarrow m(K) \leq \varepsilon/2$

$\forall x \in \mathbb{R}$ if $K_x \neq \phi$, we can find $V(x) \subset \mathbb{R}$ open s.t.

$m(V(x)) < \varepsilon/2$, $V(x) \supset K_x$

- claim: $\exists\, U(x) \ni x$ s.t. $U(x) \times V(x) \supset \pi^{-1}(U(x)) \cap K$

i.e. $\forall y \in U(x)$ want $V_x \supset K_y$

pf: suppose $\nexists\, U(x) \ni x$ with this property

$\exists\, y$ with $|y-x| < \varepsilon$