Pugh Construction

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Pugh goes by establishing the outer measure, which satisfies several axioms of an ideal measurable function. Properties of outer measure: Outer measure satisfies the following properties

(v) Empty set: true by considering [a,a] boxes.

(vi) Positivity: by definition

(vii) Monotonicity: by comparing all possible coverings and taking the inf of over the outer measures of such covering of the two sets

(viii) Finite sub-additivity: by considering the union of each covering

(x) Countable sub-additivity: same as above

(xiii) Translation invariance, consider the relationship between the translates and the translate of original covering.

every open set is a countable union of open boxes: consider rational box covering, and use definition of openness, radius of open balls

(a) If E is measurable, then $\mathbf{R}^n \setminus E$ is also measurable: by definition

(b) Translation invariant: outer measure doesn't change

(c) Boolean algebra property

(d) Every open box, and every closed box, is measurable: boxes are intersections of half planes, and half planes are measurable.

(e) Any set of outer measure zero (i.e. $m^*(E) = 0$) is measurable: properties of null set.

Countable additivity of measurable sets: show the harder direction of the inequality, consider the sup over union of disjoint covers of the set.

Hulls and kernels: consider open covers by other definition of measurability, and set $\epsilon = \frac{1}{n}$.

Product theorem: consider product of hulls and product of kernels, the two values are equal

Zero slice: consider bounded, and closed case, and show inner measure of heavy slices is zero. Then extend to unbounded case.

Lebesgue integration: undergraph of positive function.

Monotone convergence theorem: by the convergence of increasing undergraphs.

Dominated convergence: measure of undergraph of the inf of functions and the complete undergraph of sup of functions are the same.

$\int f + g = \int f + \int g$: use affine motions, meseometry

$\int \sum f = \sum \int f$: prove $\geq$ then prove $\leq$