WANG JIANZHI

MATH 105 HW 9

1. Rudin 9.6

$f(0,0) = 0$

$f(x,y) = \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$.

Firstly, we show that f is not continuous at $(0,0)$.
Pick $\varepsilon = \frac{1}{4}$. $\forall \delta > 0$, we can pick t small enough s.t. $d((t,t),(0,0)) < \delta$.

Then $|f(t,t) - f(0,0)| = \left|\frac{t^2}{t^2+t^2} - 0\right| = \frac{1}{2} > \varepsilon$. Hence, does not $\exists\ \delta > 0$ s.t. $\forall$ points p with $d(p,(0,0)) < \delta \Rightarrow |f(p) - f(0,0)| < \varepsilon$.

$\therefore$ $\boxed{f \text{ is not continuous at } (0,0)}$

$(D_1 f)(x,y) = \frac{\partial f}{\partial x}(x,y)$ $\qquad$ $(D_2 f)(x,y) = \frac{\partial f}{\partial y}(x,y)$

First consider $D_1 f(x,y)$. For $(x,y) \neq 0$, $f(x,y)$ is well defined in a neighbourhood of (x,y). $\therefore$ Suffices to use differentiation rules to get: $\frac{\partial f}{\partial x} = \frac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} = \frac{y^3 - x^2 y}{(x^2+y^2)^2}$

If $(x,y) = (0,0)$, then: $\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$

$\therefore \frac{\partial f}{\partial x}(0,0) = 0$. Hence $\boxed{\frac{\partial f}{\partial x}(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{y^3 - x^2 y}{(x^2+y^2)^2} & \text{elsewhere.} \end{cases}}$ $\therefore \frac{\partial f}{\partial x}$ exists at all points.

Similarly, if $(x,y) \neq 0$, then $f(x,y)$ well defined in a neighborhood of (x,y)
$\Rightarrow \frac{\partial f}{\partial y} = \frac{(x^2+y^2)x - xy(2y)}{(x^2+y^2)^2} = \frac{x^3 - y^2 x}{(x^2+y^2)^3}$

If $(x,y) = (0,0)$ then: $\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$

$\therefore \frac{\partial f}{\partial y}(0,0) = 0$ $\qquad$ $\therefore$ $\boxed{\frac{\partial f}{\partial y}(x,y) = \begin{cases} 0 & \text{if } (x,y) = 0 \\ \frac{x^3 - y^2 x}{(x^2+y^2)^2} & \text{elsewhere.} \end{cases}}$ $\qquad$ $\therefore \frac{\partial f}{\partial y}$ exists at all points

$\therefore$ $\boxed{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ exists at all points } p \in \mathbb{R}^2 \text{ despite } f \text{ not continuous at } (0,0).}$

2. By the assumption of the problem, $\frac{\partial f}{\partial x_i}$ exists $\forall x \in E \subset \mathbb{R}^n$.

Rudin 9.7 Hence, f must be continuous w.r.t. x_i $\forall i$.

$\frac{\partial f}{\partial x_i}$ bounded $\Rightarrow \exists M > 0$ s.t. $\forall x \in E$, $\left\| \frac{\partial f}{\partial x_i}(x) \right\| < M_i$ (Note $\frac{\partial f}{\partial x_i}(x) \in \mathbb{R}$).

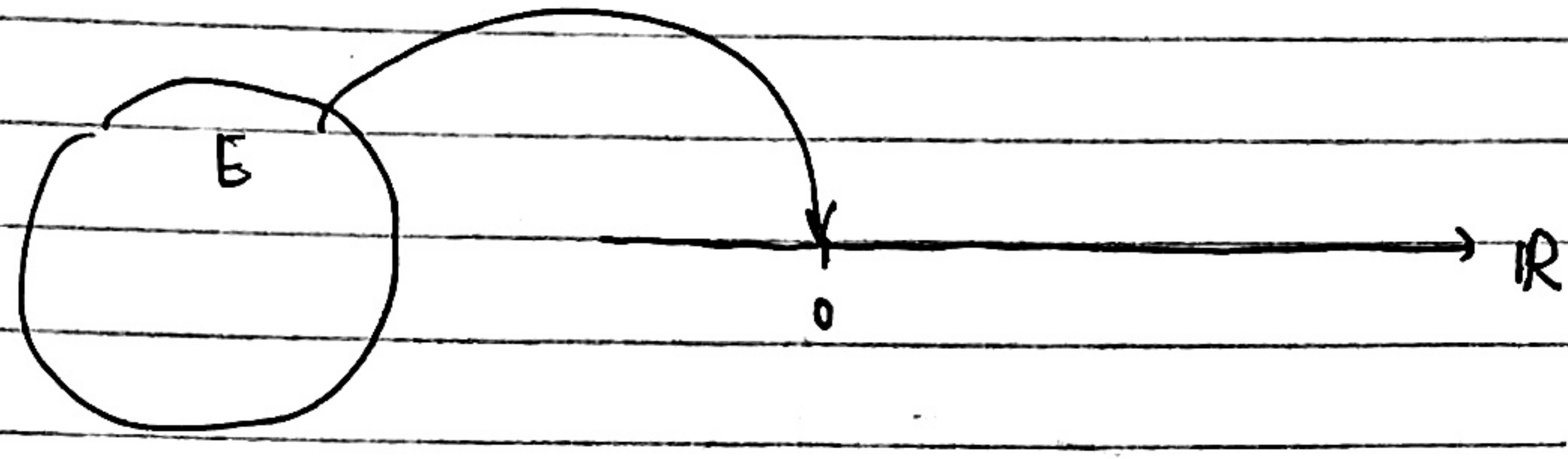
Let $\varepsilon > 0$. Pick $\delta = \frac{\varepsilon}{Mn}$ where $M = \max_i M_i$

Define $h = \sum_{i=1}^{n} h_i e_i$ s.t. $\|h\| < \delta$. Define $v_0 = 0$, $v_k = h_1 e_1 + \dots + h_k e_k$.

Then
$$\|f(x+h) - f(x)\| = \left\| \sum_{j=1}^{n} f(x+v_j) - f(x+v_{j-1}) \right\|$$
$$\leq \sum_{j=1}^{n} \| f(x+v_j) - f(x+v_{j-1}) \|$$
$$= \sum_{j=1}^{n} \left\| h_j \frac{\partial f}{\partial x_j}(x + v_{j-1} + \theta_j h_j e_j) \right\|$$
by Mean Value Theorem, where $\theta_j \in [0,$
$$= \sum_{j=1}^{n} |h_j| \left\| \frac{\partial f}{\partial x_j}(x + v_{j-1} + \theta_j h_j e_j) \right\|$$
$$\leq \sum_{j=1}^{n} |h_j| M_j \leq M \sum_{j=1}^{n} |h_j| \leq Mn \|h\| < Mn \frac{\varepsilon}{Mn} = \varepsilon.$$

Hence, $\boxed{f \text{ is continuous in } E}$ as desired.

3.



I claim that the function $f((x,y)) = d((x,y),E)$ is a continuous function $:\mathbb{R}^2 \to \mathbb{R}$ that satisfies the conditions. Note that $d((x,y),E) = \inf_{\substack{z\in E,\\ z\in\mathbb{R}^2}} \{ d((x,y),z)\}$.

Clearly, if $(x,y)\in E$, then $d((x,y),E) = 0$
Suffices to show if $(x,y)\notin E$, then $d((x,y),E) \neq 0$.

Suppose on the contrary that $\exists (x,y) \notin E$ s.t. $d((x,y),E) = 0$. Since $(x,y)\notin E$ and E is closed, then $(x,y)\in E^c$ which is open, i.e. $\exists r>0$ s.t. $B_r((x,y)) \subset E^c$. Consequently, $d((x,y),E) \geq r > 0$. Hence contradiction! $\Rightarrow d((x,y),E) > 0$ for $(x,y)\notin E$.

Next, suffices to show f continuous.

Let $(x,y)\notin E$. Then let $\varepsilon > 0$ and choose $\delta = \varepsilon$.
$d((x,y),(x',y')) < \delta$

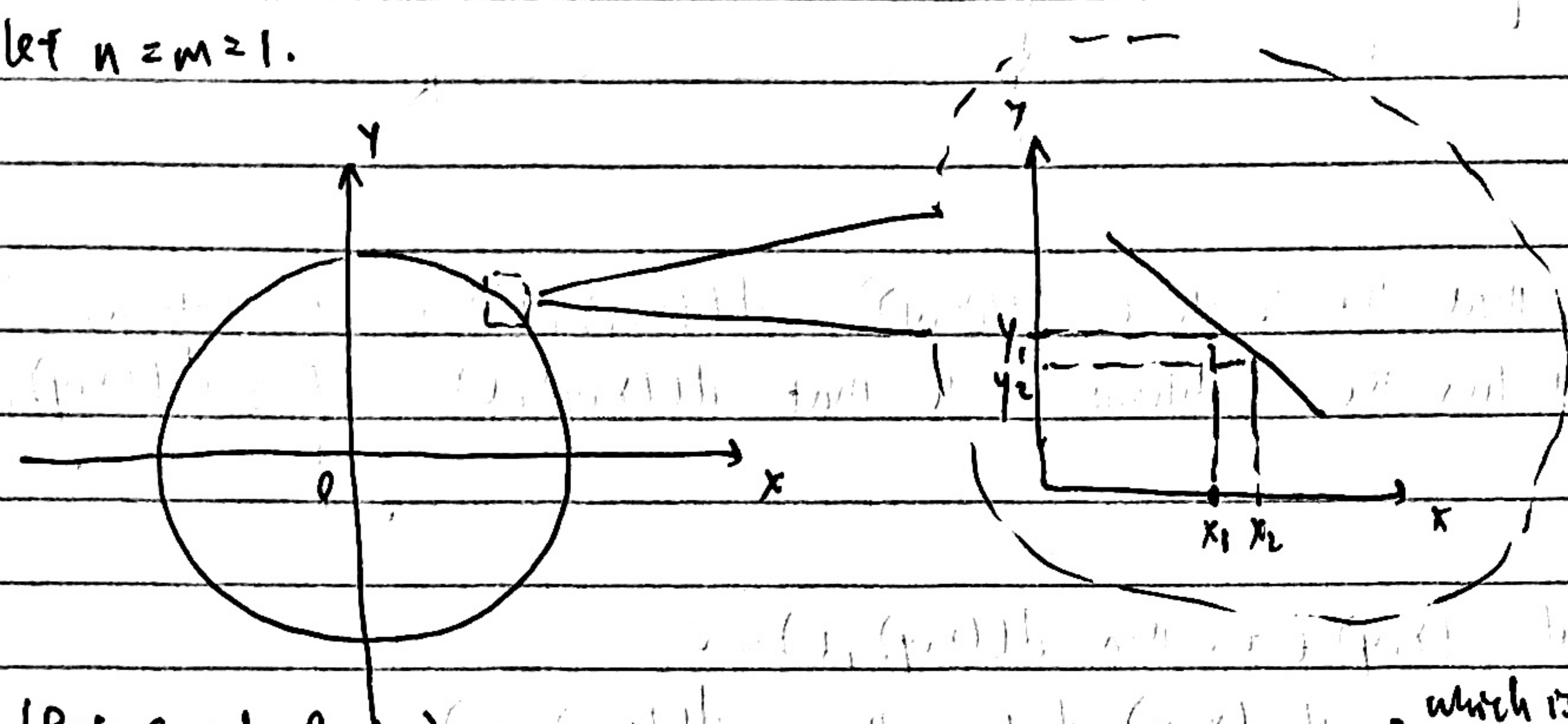
$$\Rightarrow |f(x,y) - f(x',y')| = |d((x,y),E) - d((x',y'),E)| \leq |d((x,y),(x',y'))| < \delta = \varepsilon$$

The inequality in the middle came from:

$$\left.\begin{array}{l} d((x,y),E) \overset{\Delta}{\leq} d((x',y'),E) + d((x,y),(x',y')) \\ d((x',y'),E) \overset{\Delta}{\leq} d((x,y),E) + d((x,y),(x',y')) \end{array}\right] \Rightarrow |d((x,y),E) - d((x',y'),E)| \leq d((x,y),(x',y'))$$

Hence f continuous at all $(x,y)\notin E$
Consider $(x,y)\in E$. If (x,y) is an interior point of E, then we are done since $\exists r$ s.t. $B_r((x,y)) \in E$ with $f(B_r((x,y))) = 0$. Else, $(x,y)\in \partial E$.
Let $\varepsilon>0$. Pick $\delta = \varepsilon$. Then for $d((x',y'),(x,y)) < \delta = \varepsilon$:
Then, $|f(x',y') - f(x,y)| = |f(x',y')| = \begin{cases} 0 & \text{if } (x',y')\in E \\ < \varepsilon & \text{if } (x',y') \notin E. \end{cases}$
Either way, $|f(x',y') - f(x,y)| < \varepsilon$.
$\therefore f$ is continuous for all $(x,y)\in E$. $\therefore \boxed{f \text{ is a continuous function}}$

4. let $n = m = 1$.



(Referenced Rudin)

Implicit Function Theorem is saying that locally, the function which is continuously differentiable behaves very much like its derivative. Hence, locally, we can approximate the function as a linear one and solve for y in terms of x or x in terms of y, provided $\frac{\partial f}{\partial y} \neq 0$ and $\frac{\partial f}{\partial x} \neq 0$ respectively.