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03/15/2022

MATH 105 HW 8

1. Done.

2. Let $(f_1, f_2, \dots f_n, \dots)$ be a sequence of measurable functions that converge almost everywhere to f as $k \to \infty$.

I^d refers to box in question

(a) Define $X(k,\ell) = \{x \in I^d : \forall n \geq k,\ |f_n(x) - f(x)| < \frac{1}{\ell}\}$ where d is the dimension.
Fix $\ell \in \mathbb{N}$. Since $f_n(x) \to f(x)$ for almost every x, we have: $\bigcup_k X(k,\ell) \cup Z(\ell) = I^d$.
where $Z(\ell)$ is a zero set.

Let $\varepsilon > 0$ be given. By continuity of measure, $\lim_{k\to\infty} m(X(k,\ell)) = \prod (b_i - a_i)$
we can choose indices $k_1 < k_2 < \dots$ s.t. $X_\ell = X(k_\ell, \ell)$ and $m(X_\ell^c) < \frac{\varepsilon}{2^\ell}$.
Let $X = \bigcap_\ell X_\ell$. Then $m(X^c) < \varepsilon$.
$$(\text{since } m(X^c) \leq \sum_{\ell=1}^{\infty} m(X_\ell^c) = \varepsilon).$$

Claim: f_k converges uniformly on X. Given $\sigma > 0$, choose and fix ℓ s.t. $\frac{1}{\ell} < \sigma$. Then, for all $n \geq k_\ell$, we have: $x \in X \Rightarrow x \in X_\ell = X(k_\ell, \ell) \Rightarrow |f_n(x) - f(x)| < \frac{1}{\ell} < \sigma$.

Hence, f_n converges uniformly to f less the ε-set X^c.

(b) Egoroff's Theorem remains true. Since the set S is unbounded but has finite measure, consider balls $B_r(0)$ where $r \to \infty$. $\exists M \in \mathbb{N}$ s.t. $|m(B_M(0)) - m(S)| < \frac{\varepsilon}{2}$.
Then, we can repeat the argument in part (a) for the bounded set $B_M(0)$, i.e. f_n converges uniformly to f less an $\frac{\varepsilon}{2}$-set X^c. Then, excluding $S \setminus B_M(0)$ and the $\frac{\varepsilon}{2}$ set X^c, we are done $\Rightarrow$ Egoroff's Theorem still true.

(c) Consider the moving bump functions $f_i(x) = \begin{cases} 1 & x \in [i, i+1) \\ 0 & \text{else.} \end{cases}$
Then, $(f_n)_n$ converges ~~ev~~ almost everywhere, since $\forall x$, $f_{\lceil x \rceil}(x) = 0$
$f_k(x) = 0 \ \forall k > \lceil x \rceil$.

Hence $\lim_{n\to\infty} f_n(x) = 0$. $\therefore (f_n)_n$ converges everywhere.
However, the sequence of functions is not uniformly convergent on $\mathbb{R}$.
Uniform convergence would imply $\forall x, \exists N$ s.t. $n > N$, $|f_n(x) - f(x)| < \varepsilon$. But $f = 0$
i.e. $|f_n(x)| < \varepsilon$. This is false since $f_n(x) = 1$ for $x = \lfloor n \rfloor$ hence the series is not uniformly convergent on $\mathbb{R}$.

1(d) $(f_n)_n : \mathbb{R}^n$ $\quad \varepsilon > 0.$ $\qquad m(S')<\varepsilon$ $\quad$ cube

Partition $\mathbb{R}^n$ into cubes. For each cube, $\forall \varepsilon > 0$, $\exists$ a set S' s.t. $Q \setminus S'$ will enable the sequence of functions $(f_n)_n$ to converge uniformly on it.

Let $K \subset \mathbb{R}^n$ be a compact set. Then K can be ~~expressed~~ bounded by a finite collection of cubes since K bounded. Let the cubes be $Q_1, Q_2, \dots Q_\ell$. Then, from each cube Q_i, there exists a set S_i s.t. $m(S_i) < \frac{\varepsilon}{2^i}$ $Q_i \setminus S_i$ will enable the sequence of functions $(f_n)_n$ to converge uniformly on it. Hence, let $S = \bigcup S_i$. Then $K \setminus S$ will enable the sequence of functions $(f_n)_n$ to converge uniformly, since we can take max over a finite number of cubes. i.e. $\forall \varepsilon > 0$, $\exists$ $N_1, N_2, \dots N_\ell$ s.t. $n_i > N_i \Rightarrow |f_{n_i}(x) - f(x)| < \varepsilon$. Then, set $N = \max\limits_{1 \le i \le \ell} N_i \Rightarrow n > N$ $\Rightarrow |f_n(x) - f$

Hence, the functions restricted to $K \cap S^c$ converges uniformly as desired.

3(a) $(\mathbb{R}^n, |\cdot|_1)$ normed vector space $\quad |(x_1, x_2, \dots, x_n)|_1 = \sum_i |x_i|$

$T: \mathbb{R}^n \to \mathbb{R}^n$ be linear operator, given by T_{ij}, sends x_i to y_i $\quad y_i = \sum_j T_{ij} x_j$

$$\|T\| := \sup\left\{ \frac{\|Tv\|_1}{\|v\|_1} : v \neq 0 \right\} = \sup\left\{ \|Tv\|_1 : \|v\|_1 = 1 \right\}$$

Let $y = Tv$. Then $\|y\| = \sum_i |y_i| = \sum_i \left|\sum_j T_{ij} v_j\right|$

$$\Rightarrow \sup_{v: \|v\|_1 = 1} \|y\| = \sup_{v: \|v\|_1 = 1} \sum_i \left|\sum_j T_{ij} v_j\right| = \sup \sum_i \left|\sum_j T_{ij} v_j\right| \quad \text{subject to } |v_1| + |v_2| + \dots + |v_n| = 1$$

$$= \sup \sum_i |t_i^T v| \qquad T = \begin{bmatrix} - t_1^T - \\ - t_2^T - \\ \vdots \\ - t_n^T - \end{bmatrix}$$

For me, I think of it geometrically, i.e. the sup is achieved at one of the vertices of the ℓ_1 ball, i.e. when $v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ or $\dots$ $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.
This is because at every step, the $\sum_i |t_i^T v|$ is just a system of linear equalities. Hence, if we take derivative with respect to any component, it will do no worse by shifting that component towards 0 or 1 depending on the sign of the derivative.
Hence, this problem is equivalent to $\sup \|Tv\|_1$ for $v \in \left\{ \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$

$= \sup \|t_i\|_1$ where t_i is the ith row of T

$\therefore$ $\boxed{\|T\| = \sup \|t_i\|_1 = \max \|t_i\|_1 \text{ (i.e. max } \ell_1 \text{ norm of column)}}$

Another solution is by algebra:

$$\|Tv\|_1 = \left\| \begin{bmatrix} | & | & & | \\ t_1 & t_2 & \cdots & t_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|_1 = \left\| v_1 \begin{bmatrix} | \\ t_1 \\ | \end{bmatrix} + v_2 \begin{bmatrix} | \\ t_2 \\ | \end{bmatrix} + \cdots + v_n \begin{bmatrix} | \\ t_n \\ | \end{bmatrix} \right\|_1$$

$$\leq \left\| v_1 \begin{bmatrix} | \\ t_1 \\ | \end{bmatrix} \right\|_1 + \left\| v_2 \begin{bmatrix} | \\ t_2 \\ | \end{bmatrix} \right\|_1 + \cdots + \left\| v_n \begin{bmatrix} | \\ t_n \\ | \end{bmatrix} \right\|_1$$

$$= |v_1| \left\| \begin{bmatrix} | \\ t_1 \\ | \end{bmatrix} \right\|_1 + |v_2| \left\| \begin{bmatrix} | \\ t_2 \\ | \end{bmatrix} \right\|_1 + \cdots + |v_n| \left\| \begin{bmatrix} | \\ t_n \\ | \end{bmatrix} \right\|_1$$

$$\leq \left(|v_1| + |v_2| + \cdots + |v_n|\right) \max\left(\left\| \begin{bmatrix} | \\ t_1 \\ | \end{bmatrix} \right\|_1, \cdots, \left\| \begin{bmatrix} | \\ t_n \\ | \end{bmatrix} \right\|_1 \right)$$

$$= \|v\|_1 \cdot \max\left(\left\| \begin{bmatrix} | \\ t_1 \\ | \end{bmatrix} \right\|_1, \cdots, \left\| \begin{bmatrix} | \\ t_n \\ | \end{bmatrix} \right\|_1 \right)$$

$$= \max\left(\left\| \begin{bmatrix} | \\ t_1 \\ | \end{bmatrix} \right\|_1, \cdots, \left\| \begin{bmatrix} | \\ t_n \\ | \end{bmatrix} \right\|_1 \right).$$

Note that this max is achievable, i.e. when
$v = t_{max}$ where $t_{max} = \arg\max\left(\left\| \begin{bmatrix} | \\ t_1 \\ | \end{bmatrix} \right\|_1, \cdots, \left\| \begin{bmatrix} | \\ t_n \\ | \end{bmatrix} \right\|_1 \right)$

Hence $\boxed{\|T\|_1 = \max \|t_i\|}$ as desired. $\boxed{(\text{max of } l_1 \text{ norms of columns})}$

3(b) $\|T\|_\infty := \sup\left\{ \dfrac{\|Tv\|_\infty}{\|v\|_\infty} : v \neq 0 \right\}$, $\quad \|v\|_\infty = \max\limits_{1 \leq i \leq n} |v_i|$.

$$= \sup\left\{ \|Tv\|_\infty : \|v\|_\infty = 1 \right\}$$

$$\|Tv\|_\infty = \left\| \begin{bmatrix} - & t_1^T & - \\ - & t_2^T & - \\ & \vdots & \\ - & t_n^T & - \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|_\infty = \max\left(|t_1^T v|, |t_2^T v|, \cdots, |t_n^T v| \right)$$

Note by Cauchy Schwarz, actually Hölder's inequality, $|t_1^T v| \leq \|t_1\|_1 \|v\|_\infty$
$= \|t_1\|_1$

Hence $\|Tv\|_\infty = \max\left(|t_1^T v|, |t_2^T v|, \ldots, |t_n^T v| \right)$
$\leq \max\left(\|t_1\|_1, \|t_2\|_1, \ldots, \|t_n\|_1 \right)$

Note that equality is achievable when $v_i = \text{sign}(t_{ij})$

Hence, $\boxed{\begin{aligned} \|T\|_\infty &= \max \|t_i\|_1 \\ &= \max\ l_1 \text{ norm of rows.} \end{aligned}}$

4. Hölder's inequality: For conjugates p, q s.t. $\frac{1}{q} + \frac{1}{p} = 1$, $p, q \geq 1$.

$$\left(\sum_{i=1}^{n} |x_i y_i|\right) \leq \left(\sum |x_i|^p\right)^{\frac{1}{p}} \left(\sum |y_i|^q\right)^{\frac{1}{q}}$$

i.e.

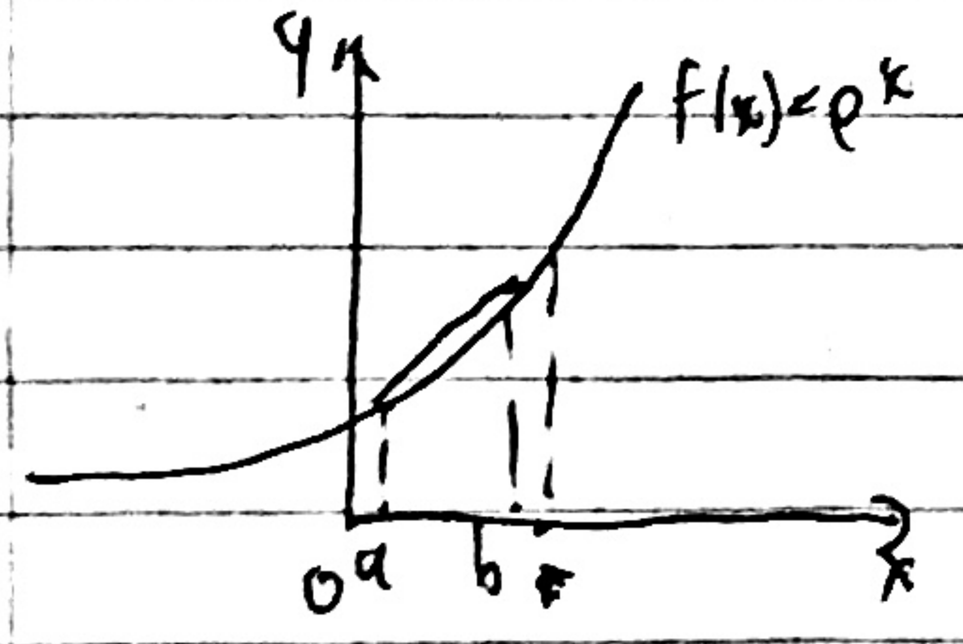
$$|x^T y| \leq \|x\|_p \|y\|_q$$

Firstly, we prove the Young's inequality: (Most parts below are referenced from UPENN MATH ~brweber/courses/2011/Math361/Notes/YMandH.pdf)

Young's Inequality

When $1 < p < \infty$, $a, b \geq 0$,
Young's inequality gives: $ab \leq \frac{p-1}{p} a^{\frac{p}{p-1}} + \frac{1}{p} b^p$

Lemma: For $t \in [0,1]$, then $e^{ta + (1-t)b} \leq te^a + (1-t)e^b$
This is due to the secant line trick in Evan Chen's notes.



Because $f(x) = e^x$ is convex, the line joining (a, e^a) and (b, e^b) lies above the curve

$\Rightarrow \forall t \in [0,1], \; f(ta + (1-t)b) \leq t f(a) + (1-t) f(b)$

$\Rightarrow \boxed{e^{ta + (1-t)b} \leq te^a + (1-t)e^b}$

Proof: (Young) $\text{LHS} = ab = e^{\log a + \log b} = e^{\underbrace{\frac{p-1}{p}}_{t} \frac{p}{p-1} \log a + \underbrace{\left(1 - \frac{p-1}{p}\right)}_{1-t} \left(\frac{1}{1 - \frac{p-1}{p}}\right) \log b}$

$$\leq \frac{p-1}{p} e^{\frac{p}{p-1} \log a} + \left(1 - \frac{p-1}{p}\right) e^{\frac{1}{1-\frac{p-1}{p}} \log b}$$

$$= \frac{p-1}{p} a^{\frac{p}{p-1}} + \frac{1}{p} b^p = \text{RHS}. \quad \boxed{\text{as desired}}$$

Hölder's

Proof (Hölder's) $\sum_{i=1}^{\infty} x_i y_i \leq \|X\|_{\cancel{p}q} \|Y\|_p$.

$\text{LHS} = \sum_{i=1}^{\infty} \frac{x_i}{\|X\|_q} \frac{y_i}{\|y\|_p} \leq \sum_{i=1}^{\infty} \frac{|x_i|}{\|x\|_q} \frac{|y_i|}{\|y\|_p}$ (might as well make both positive to prove a tighter bound)

$$\leq \sum_{i=1}^{n} \frac{p-1}{p} \cancel{\frac{|x_i|}{\|x\|}} \frac{|x_i|^q}{\|x\|_q^q} + \frac{1}{p} \frac{|y_i|^p}{\|y\|_p^p}$$

(because by Young's inequality, $\underbrace{\frac{|x_i|}{\|x\|_q}}_{a} \cdot \underbrace{\frac{|y_i|}{\|y\|_p}}_{b} \leq \frac{p-1}{p} \cdot \left(\frac{|x_i|}{\|x\|_q}\right)^{\overbrace{\frac{p}{p-1}}^{= q!}} + \frac{1}{p}\left(\frac{|y_i|}{\|y\|_p}\right)^p$

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(Continued)

$$= \sum_{i=1}^{n}\left(\frac{p-1}{p}\frac{|x_i|^q}{\|x\|_q^q} + \frac{1}{p}\frac{|y_i|^p}{\|y\|_p^p}\right)$$

$$= \frac{p-1}{p}\frac{1}{\|x\|_{\frac{p}{p-1}}^{\frac{p}{p-1}}}\sum_{i=1}^{\infty}|x_i|^{\frac{p}{p-1}} + \frac{1}{p}\frac{1}{\|y\|_p^p}\sum_{i=1}^{\infty}|y_i|^p$$

$$= \frac{p-1}{p}\cdot\frac{\|x\|_q^q}{\|x\|_q^q} + \frac{1}{p}\frac{\|y\|_p^p}{\|y\|_p^p} = 1$$

$$\therefore \sum_{i=1}^{\infty}\frac{x_i}{\|x\|_{\frac{p}{p-1}}}\frac{y_i}{\|y\|_p} \leq 1 \Rightarrow \boxed{\sum_{i=1}^{\infty} x_i y_i \leq \|x\|_q\|y\|_p}$$ as desired.

Minkowski Theorem

If $1 \leq p < \infty$, whenever $x, y \in V_p$ we have $\|x+y\|_p \leq \|x\|_p + \|y\|_p$.

$$\left(\sum_{i=1}^{\infty}|x_i+y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum|x_i|^p\right)^{\frac{1}{p}} + \left(\sum|y_i|^p\right)^{\frac{1}{p}}$$

I followed Bath.ac.uk/masmdp/measure.dir.bho/notesp54to61.pdf.

$\|f+g\|_p \leq \|f\|_p + \|g\|_p$. Suppose $f, g \geq 0$ pointwise. If $\|f+g\|_p = 0$, then $f+g=0$ a.e.

$\Rightarrow \|f+g\|_p = 0 \leq \|f\|_p + \|g\|_p = 0$ ✓.

Else, $\|f+g\|_p > 0$.

Since $x \to x^p$ convex, $\left(\frac{f+g}{2}\right)^p \leq \frac{1}{2}(f^p+g^p)$ pointwise

$$(f+g)^p = f(f+g)^{p-1} + g(f+g)^{p-1}$$

$$\int f(f+g)^{p-1} \leq \|f\|_p\left(\int (f+g)^{(p-1)q}\right)^{\frac{1}{q}} = \|f\|_p\left(\int (f+g)^p\right)^{\frac{1}{q}}$$

Similarly,

$$\int g(f+g)^{p-1} \leq \|g\|_p\left(\int (f+g)^p\right)^{\frac{1}{q}}$$

$$\therefore \int (f+g)^p \leq \left[\|f\|_p + \|g\|_p\right]\left(\int (f+g)^p\right)^{\frac{1}{q}}$$

$$\Rightarrow \left(\int (f+g)^p\right)^{1-\frac{1}{q} \longrightarrow \frac{1}{p}} \leq \|f\|_p + \|g\|_p \Rightarrow \boxed{\|f+g\|_p \leq \|f\|_p + \|g\|_p}$$