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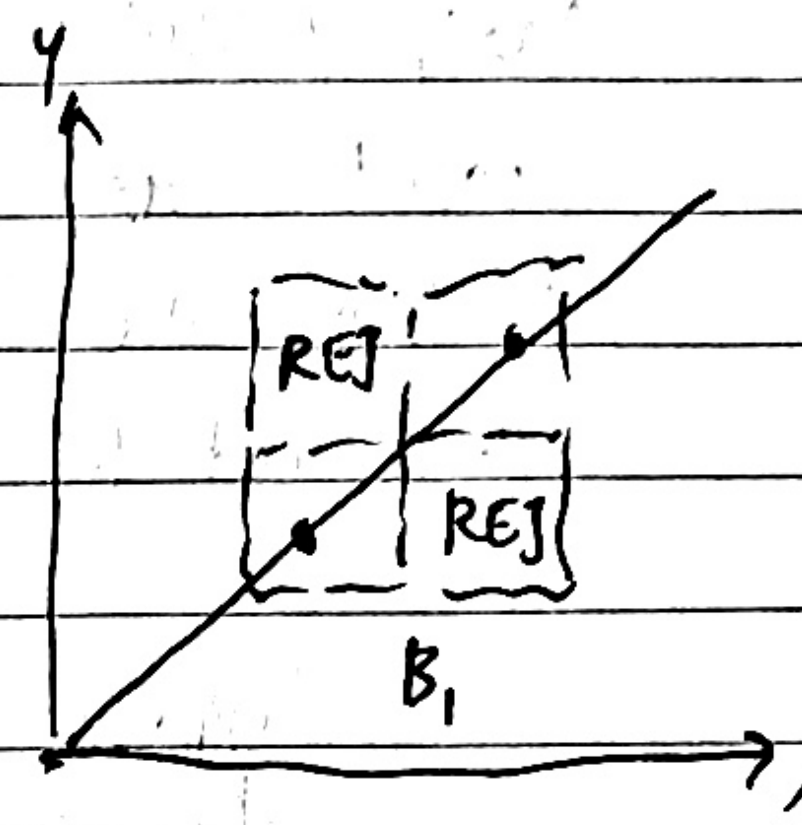
MATH 105 HW 3

0. Done



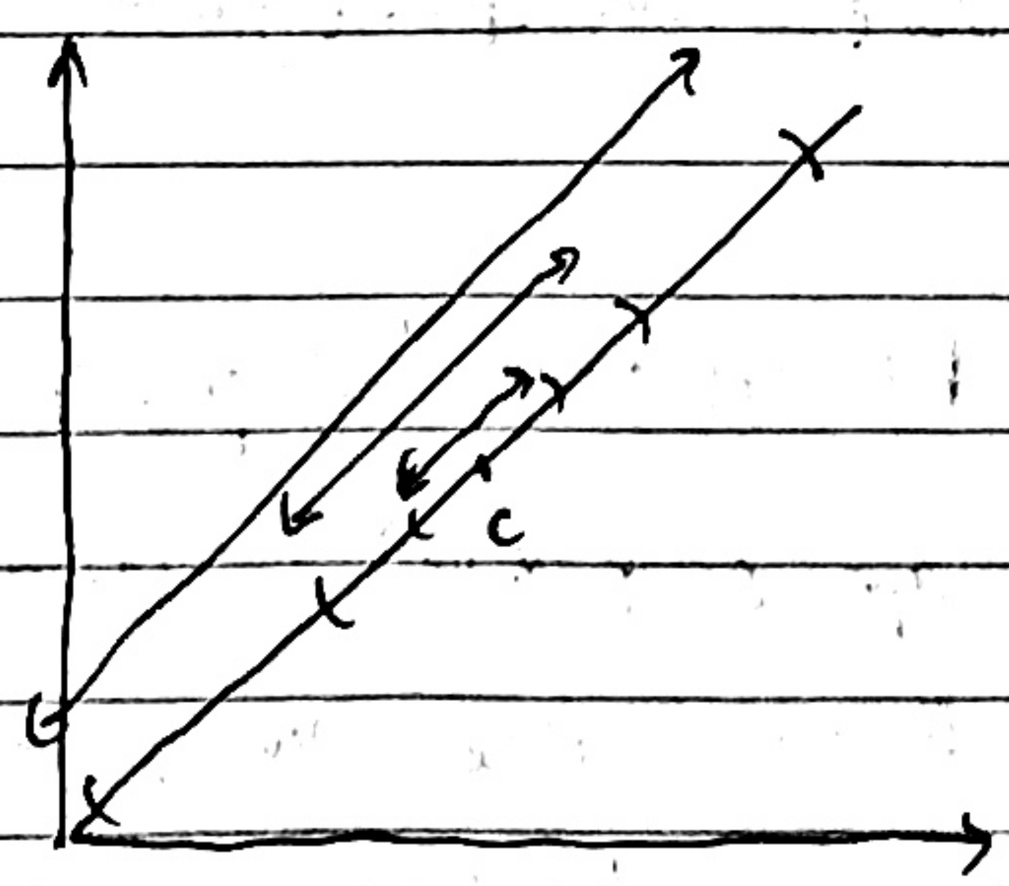
[Punch 3] 1. Firstly, I will prove the special case $\{y=x\}$. Consider a line segment first. Consider a box B_1 open that covers this line segment.

Now, we divide B_1 into four boxes, reject the top-left and bottom right ones, and extend the remaining two boxes slightly so that they cover any middle points, as illustrated in the diagram. We can do this such that the total volume of the current boxes is $\leq \frac{2}{3}|B_1|$.

Note that we can continue repeating this algorithm and this will still give us a finite cover of the line segment with total volume $\leq \left(\frac{2}{3}\right)^n |B_1|$ after n iterations.

$\therefore$ $m^*(\text{line segment}) \leq \left(\frac{2}{3}\right)^n |B_1| \quad \forall n \Rightarrow \boxed{m^*(\text{line segment}) = 0}$

For an infinitely long segment, we can treat it as follows: we pick a center c.



Consider the line segments $[c-2^n, c+2^n]$ with 2^n being the length of the segment.

for each $n \in \mathbb{N}$, we can cover the line with boxes of total volume $< \frac{\varepsilon}{2^n}$.

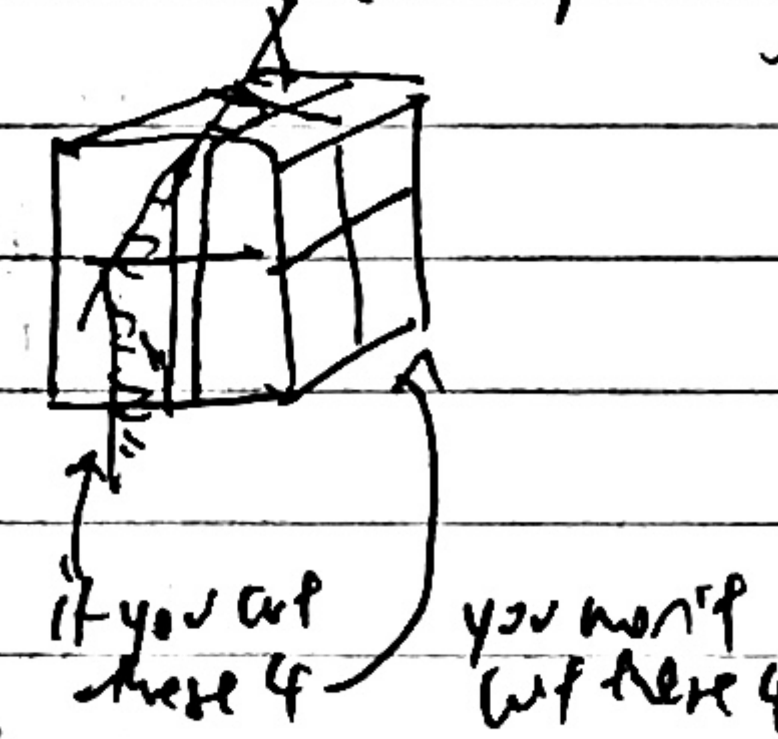
Hence, the entire line segment can be covered by boxes of volume $< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \dots = \varepsilon$

Since ε is arbitrarily chosen, the entire diagonal line has outer measure 0 and thus is a zero set.

next page for generalization.

Now, consider a general n-dimension plane ~~of~~ that belongs in dimension $< n$.
Consider an open box B that has a nonempty intersection with this plane. Partition the box into 2^n smaller boxes (2 for each dimension). The plane does not intersect all 2^n of these smaller boxes.

Suppose otherwise that the plane intersects all of these 2^n boxes, pick a dimension. The plane intersects all 2^{n-1} boxes in the $n-1$ other dimensions
$\Rightarrow$ the plane must be restricted to ~~that~~ a range in that dimension, meaning it will not intersect the other 2^{n-1} boxes.



Hence, the total volume of required boxes drop by at least a factor of

$$\underbrace{\frac{2^n - 1 + \frac{1}{2}}{2^n}}\text{, for safety}$$

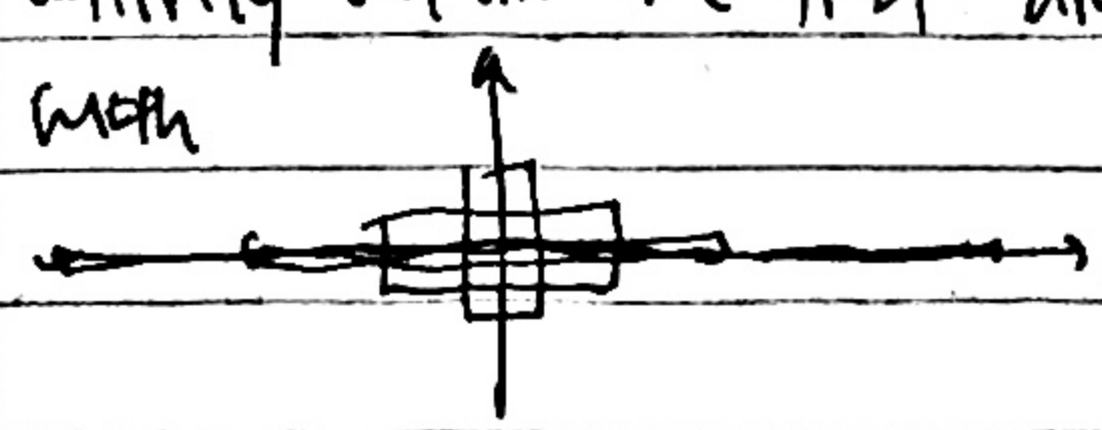
$$= \frac{2^n - \frac{1}{2}}{2^n} < 1$$

After many iterations of this algorithm, we still produce a covering of the plane, and $m^*(\text{plane}) < \left(\frac{2^n - \frac{1}{2}}{2^n}\right)^k$ where k is the number of iterations.

Since $\lim_{k \to \infty} \left(\frac{2^n - \frac{1}{2}}{2^n}\right)^k = 0$, we are done. $m^*(\text{plane}) = 0$.

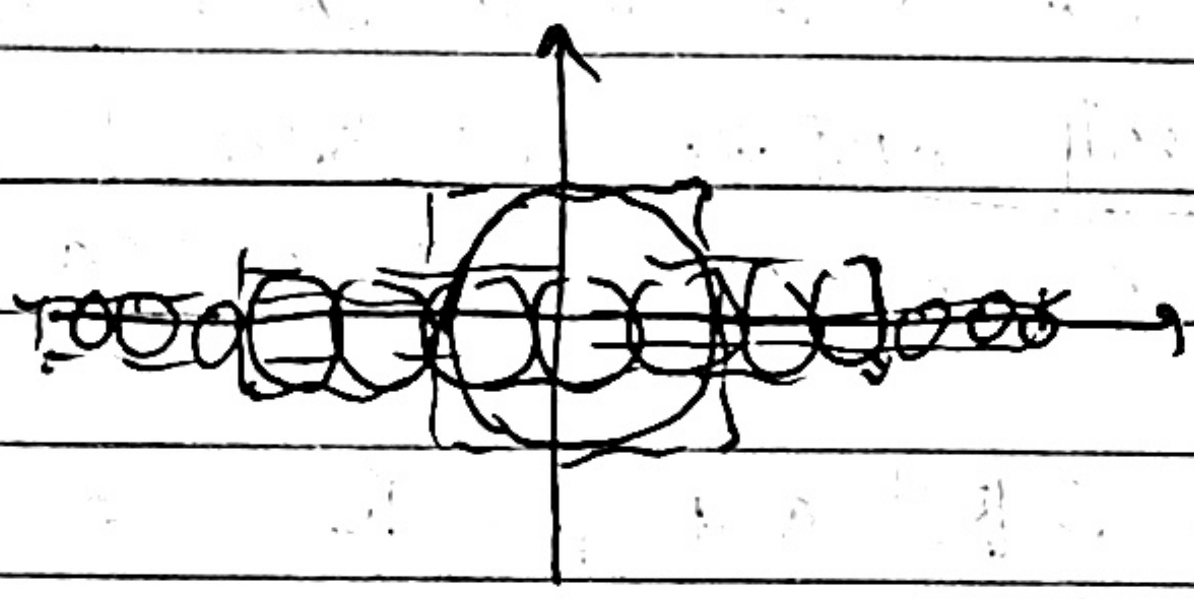
Two other ways which I believe will work, but not from first principles.

① By zero slice theorem, let E denote the plane $\subset \mathbb{R} \times \mathbb{R}^{n-1}$. Clearly, a plane is closed (the complement is open) and hence is measurable. Thus, if the plane lies entirely within the first dimension $\mathbb{R}$, then it is a hyperplane and we can solve it with



Else, every slice of E is a zero set, since every slice of E with the plane ~~at~~ in ~~a~~ $n-1$ dimensional plane which has measure 0 by induction.
$\Rightarrow$ $\boxed{E = \text{plane is a measure } 0 \text{ set}}$

1. Balls are open and can therefore be treated as countable union of ~~cbe~~ open cubes and a measure 0 set. Note that balls are invariant to rotations etc, and we can transform our arguments for hyperplane has a measure 0 to using balls



Hence, for arbitrary direction, we can just consider the translation of these balls (equivalently the open boxes that make up these balls).
$\therefore$ measure of an affine plane $= 0$.

[Pugh 6] 2. 16. Lemma: Every open set in n-space is a countable disjoint union of open cubes plus a zero set.

For n-dimension, let $U \subset \mathbb{R}^n$ be open. Then consider the intersection of U with $B_0(1)$, $B_0(2) \setminus \overline{B_0(1)}$, ..., $B_0(n) \setminus \overline{B_0(n-1)}$, ...

All of these are open, and the intersection of U with them are open. Those that lie on the surface of $B_0(z)$, $z \in \mathbb{Z}^+$ form a zero set and need not be considered. Denote $B_i' = \begin{cases} B_0(1) & \text{if } i=1 \\ B_0(i) \setminus \overline{B_0(i-1)} & \text{if } i>1 \end{cases}$

Consider $U \cap B_i'$ separately. Each of these can be expressed as a countable disjoint union of open cubes plus a zero set by using the same algorithm:

Partition the space into disjoint cubes $[0,1]^n$, accept all those that completely lies in $U \cap B_i'$ and reject all those that do not ~~lie completely~~ intersect $U \cap B_i'$. at all. For those that intersect both $U \cap B_i'$ and $(U \cap B_i')^c$, we further subdivide them into 2^n smaller cubes and repeat the algorithm

Since $U \cap B_i'$ is open, $\forall x \in U \cap B_i'$, $\exists r$ s.t. $B_x(r) \subset (U \cap B_i')$
if any of x's coordinates is an integer multiple of $\frac{1}{2^k}$ for some k, we do not consider it, for it will form part of the measure 0 set. Else, $\exists l$ s.t. $\frac{1}{2^l} < \frac{r}{\sqrt{n}}$. then x will be contained in a box of size $\left[0, \frac{1}{2^l}\right]^n$.

[next page]

Hence, each of $U \cap B_i'$ can be expressed as a countable disjoint union of open cubes plus a zero set.

Since countable addition of a zero set is still a zero set, and countable union of countable unions is still countable, we have U can be expressed as a $\boxed{\text{countable disjoint union of open cubes plus a zero set.}}$

21: Measurable Product Theorem: If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^k$ are measurable, then $A \times B$ is measurable and $m(A \times B) = m(A) \cdot m(B)$.

Since the zero sets are already proved, assume $m(A), m(B) \neq 0$.

Wait, firstly, need to show lemma 23, lemma 24 holds for higher dimensions as well.

For lemma 23, Suppose $m(A) = 0$. Then, we can cover A with open intervals I_i with length $< \frac{\varepsilon}{2^i (2\lambda)^k}$ where k is the dimension of B. Then the volume of $I_i \times [-\lambda, \lambda]^k \supset B < \frac{\varepsilon}{2^i}$. Then the union of all of these rectangles cover $A \times \mathbb{R}^k$ and has measure $< \varepsilon$ $\forall \varepsilon$ $\therefore$ $A \times B$ is a zero set when A is a zero set.

For lemma 24, $U \times V$ is open $\Rightarrow$ measurable. By lemma 16, which we proved already for higher dimensions, $U = \bigsqcup_i B_i \cup Z_U$ and $V = \bigsqcup_j C_j \cup Z_V$ where B_i, C_j are boxes and Z_U, Z_V are measure 0 sets.

$U \times V = \bigsqcup_{i,j} B_i \times C_j \cup Z$, a measure 0 set by lemma 23, which we proved in higher dimensions

Since $\left(\sum_i m(B_i)\right)\left(\sum_j m(C_j)\right) = \sum_{ij} m(B_i) m(C_j) = \sum_{i,j} m(B_i \times C_j)$

we have $\boxed{m(U \times V) = mU \cdot mV.}$

Back to the actual theorem, the claim that the hull / kernel of a product is the product of hull/kernel still holds true (did not use lower dimension arguments).

we can proceed by partitioning the space into $[0,1]^n$ cubes for A and $[0,1]^k$ cubes for B. we apply the same arguments: U_n, V_n be sequences of open sets in I^n, I^k converging down to H_A, H_B, then $U_n \times V_n$ is a sequence of open sets in I^{n+k} converging down to $H_A \times H_B$.

Applying downward measure continuity, $m(U_n \times V_n) \to m(H_A \times H_B)$. By lemma 24, we have $m(U_n \times V_n) = m(U_n) \cdot m(V_n)$ which $\to m(A), m(B)$ respectively. Hence, $\boxed{m(A \times B) = m(A) \cdot m(B)}$

[Pugh 11] 3. Firstly, I will prove Pugh 11 (c). Note $m^*(A) \le J^*(A)$ because $J^*(A)$ requires
12 a finite covering of A with open boxes, while $m^*(A)$ just requires countable. Due to the increased restriction, $m^*(A) \le J^*(A)$ (taking infimum over a smaller set).

If A is compact, by Heine Borel's theorem, any open cover of A has a finite subcover. Hence, for any countable open cover of A, we can achieve the same, if not better for $J^*(A)$. Hence $m^*(A) \ge J^*(A)$. Combined with the above inequality $\boxed{m^*(A) = J^*(A) \text{ if } A \text{ compact}}$

Now, moving onto 12, clearly from 11(c), since A is assumed to be a bounded set, $\bar{A}$ is compact $\Rightarrow \boxed{J^*\bar{A} = m\bar{A}}$

Clearly, we also have $J^*A \le J^*\bar{A}$ by monotonicity of J.
Suffices to show $J^*A + \varepsilon \ge J^*\bar{A}$ $\forall \varepsilon$.
Let $B_1, \dots B_N$ be the open intervals s.t. $\bigcup B_i$ covers A and $\sum |B_i| < J^*A + \frac{\varepsilon}{2}$.
Enlarge each B_i in all dimensions s.t. their volume becomes $|B_i| + \frac{\varepsilon}{2 \cdot 2^i}$
i.e. all dimensions of B_i increase by at least a positive constant $\delta_i > 0$.
I claim that this new sequence $B_1', B_2', \dots B_N'$ covers $\bar{A}$.

Suppose otherwise, then $\exists x \in \bar{A}$ that is not covered. ~~This means $\forall r > 0$~~, since $x \in \bar{A}$, ~~it means $\forall r >$~~ x definitely cannot $\in A$ since the boxes $\bigcup B_i$ already cover A and B_i' are an enlargement of the boxes

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Hence $x \in \bar{A} \setminus A \Rightarrow \forall n > 0,\ \exists\ x_n$ s.t. $x_n \in B_x(\frac{1}{n})$ that is covered $\in \mathbb{Z}$
(since x is on the boundary and has non empty intersection with A).

~~Consider this sequence (x_n). Since $\bar{A}$ compact, $\exists$ subsequence $(x_{n_k})_k$ that converges~~

In particular, $\exists\ x_i$ s.t. $\frac{1}{i} < \underbrace{\frac{\min(\delta_1, \delta_2, \dots \delta_i)}{\sqrt{n}}}$ enlargements of the boxes

$\sqrt{n}$ ← dimension of the space

s.t. x_i is covered in a box, say B_i'

Thus, the enlargement of box would have covered x, as desired. (contradiction!) Since we assumed x is not covered.

Hence $J^* A + \varepsilon \geq J^* \bar{A}$

$\therefore \boxed{J^* A = J^* \bar{A} = m\bar{A}}$