MATH 105 HW2

Definition A subset E is measurable if $\forall \varepsilon > 0$, $\exists$ open set $U \supseteq E$ s.t. $m^*(U \setminus E) < \varepsilon$.

Lemma 0 By definition of outer measure, $m^*(\Omega) = \inf\{\sum |B_i|,\ \Omega \subset \bigcup_i B_i,\ B_i \subset \mathbb{R}^n \text{ open boxes}\}$
By finite subadditivity, $m^*(E \cup F) \leq m^*(E) + m^*(F)$
Suffices to show $m^*(E \cup F) \geq m^*(E) + m^*(F)$

Consider $\{B_i\}$ to be a countable sequence of open boxes that cover $E \cup F$
let $\delta = \text{dist}(E, F) = \inf\{|x-y| : x \in E, y \in F\}$.

For any open box B_i, we split it into a finite number of open boxes with every dimension $\leq \frac{\delta}{2\sqrt{d}}$ and open planes $\in \mathbb{R}^k$ $(k \leq d-1)$ (which has outer measure 0). The algorithm proceeds as follow: For each dimension of $B_i \geq \frac{\delta}{\sqrt{d}}$ split it into multiples of $\frac{\delta}{2\sqrt{d}}$ with remainder. Use planes to cover the split points/planes. Thus, any point $\in B_i$ will $\in$ this finite list of small boxes and planes.

By the splitting of one box into finite smaller boxes and planes, the outer measure remains the same. Hence, suffices to consider this new sequence of boxes $\{B_i'\}$ where every dimension of $B_i \leq \frac{\delta}{2\sqrt{d}}$.

For each i, there cannot be the case where $e \in E$, $f \in F$ but $e, f \in B_i'$.
because $|e-f| \geq \inf\{|x-y| : x \in E, y \in F\} \geq \frac{\delta}{2} \geq \sqrt{d} \cdot \frac{\delta}{2\sqrt{d}}$
where $\sqrt{n} \cdot \frac{\delta}{2\sqrt{d}}$ is the distance between furthest points in B_i'.

Hence, we construct two (disjoint) subsequences $\{B_i^{(1)}\}, \{B_i^{(2)}\}$ where $B_i \in (1)$ if $\exists e \in E$ s.t. $e \in B_i$ and $B_i \in (2)$ if $\exists f \in F$ s.t. $f \in B_i$.
Since these are subsequences that cover E and F respectively.
$m^*(E \cup F) \geq m^*(E) + m^*(F)$ since for any open box covering of $E \cup F$ covers E and F too.

$\therefore \boxed{m^*(E \cup F) = m^*(E) + m^*(F)}$ as desired.

lemma 1 By definition of outer measure, $m^*(A) = \inf\{\sum |B_i| \mid A \subset \cup B_i,\ B_i \text{ open}\}$.

Clearly,

$$m^*(A) \geq \inf\{m^*(U) \mid A \subset U,\ U \text{ open}\}$$

This is because $\cup_i B_i$ is open since it is the countable union of open boxes.
Hence, RHS is smaller simply because it is the infimum of a larger set.
and $m^*(\cup_i B_i) \leq \sum m^*(B_i) = \sum |B_i|$

Suffices to show $m^*(A) \leq \inf\{m^*(U) \mid U \supset A,\ U \text{ open}\}$

I claim that every open set $U \in \mathbb{R}^n$ can be written as the countable union of open boxes and planes.

Consider open boxes of the form $(0,1)^n, (0,\frac{1}{2})^n, (0,\frac{1}{4})^n, \dots (0,\frac{1}{2^k})^n$
and their translations by $1, \frac{1}{2}, \dots, \frac{1}{2^k}$ respectively.
Starting from $(0,1)^n$, if it or any of its translations by integer multiple of 1 $\subset U$, then add it to the set B_{2^0}. proceed to $(0,\frac{1}{2})^n$. If it or any of its translations $\subset U$, then add it to the set B_{2^1} and so on.

$B_{2^0}, B_{2^1}, \dots$ are all countable sequences of open boxes, so their union is countable. Note that we also have to consider open planes to include the split points ~~of~~ at $1, \frac{1}{2}, \frac{1}{4}, \dots$

For any point $x \in U$, $B_r(x) \subset U$ for some $r > 0$. and $r > \frac{1}{2^k}$ for some $k \in \mathbb{Z}$
Hence $x \in$ a box in B_{2^k} $\Rightarrow x \in \cup_i B_{2^i}$. Otherwise, $x \in$ plane.

We know $U \supset \cup_i B_i$ and $m^*(\cup_i B_i) \leq m^*(U) \leq m^*(\cup_i B_i) + \overset{0}{\overset{\shortparallel}{m^*(\text{planes})}}$
$m^*(\cup_i B_i) \leq m^*(U) \leq$ $\cup$ planes $\Rightarrow m^*(\cup_i B_i) = m^*(U)$
Hence $m^*(U)$ can be expressed as $m^*(\cup_i B_i \cup \text{planes})$ which is $\sum |B_i|$ so
$m^*(U)$ can be expressed as $\sum |B_i|$ $\cup$ planes where $A \subset \cup B_i \cup$ planes

Hence $\boxed{m^*(A) = \inf\{m^*(U) \mid U \supset A,\ U \text{ open}\}}$

* Planes can just be regarded as open boxes with arbitrarily small outer measure

Lemma 2 Let $\varepsilon > 0$.

E_i is measurable $\Rightarrow \exists U_i$ s.t. U_i open and $U_i \supset E_i$ and $m^*(U_i \setminus E_i) < \frac{\varepsilon}{2^i}$

Then, consider the open cover $\bigcup_i U_i$.

Let $x \in (\bigcup_i U_i) \setminus (\bigcup_i E_i)$

* Then $x \in U_j \setminus E_j$ for some $j \in \mathbb{N}$

Hence $(\bigcup_i U_i) \setminus (\bigcup_i E_i) \subset \bigcup (U_i \setminus E_i)$

$$\Rightarrow m^*((\bigcup_i U_i) \setminus (\bigcup_i E_i)) \overset{\text{monotonicity}}{\leq} m^*(\bigcup (U_i \setminus E_i))$$
$$\leq \sum m^*(U_i \setminus E_i)$$
$$< \sum \frac{\varepsilon}{2^i} = \varepsilon$$

Since ε is arbitrarily chosen, $\bigcup_i E_i$ is measurable by definition.

Lemma 3. Consider the open balls and their difference (i.e. $\{x \mid z \leq |x| \leq z+1, z \in \mathbb{Z}\}$
These sets are closed, since their complement is open.
The intersection of these sets with A (a closed set) is closed.
Hence A can be written as the countable union of bounded closed subsets
(the intersection with the closed differences of unit balls).

$\therefore$ Suffices to prove any bounded closed (compact) set A is measurable.

At this step, I do not know how to proceed, so I read Princeton's measure theory notes (assets.press.princeton.edu/chapters/s8008.pdf) and filled in some steps.
Let F be our compact set.
Let $\varepsilon > 0$. By Lemma 1, $m^*(F) = \inf\{m^*(U) \mid F \subset U, U \text{ open}\}$
In particular, $\exists$ open set U s.t. $F \subset U$ and $m^*(F) + \varepsilon \geq m^*(U)$
(since $m^*(F) + \varepsilon$ is not a lower bound)

Since F closed, $U - F$ is open. (because $\mathbb{R}^n \setminus F$ open and thus $U - F = U \cap (\mathbb{R}^n \setminus F)$ open)

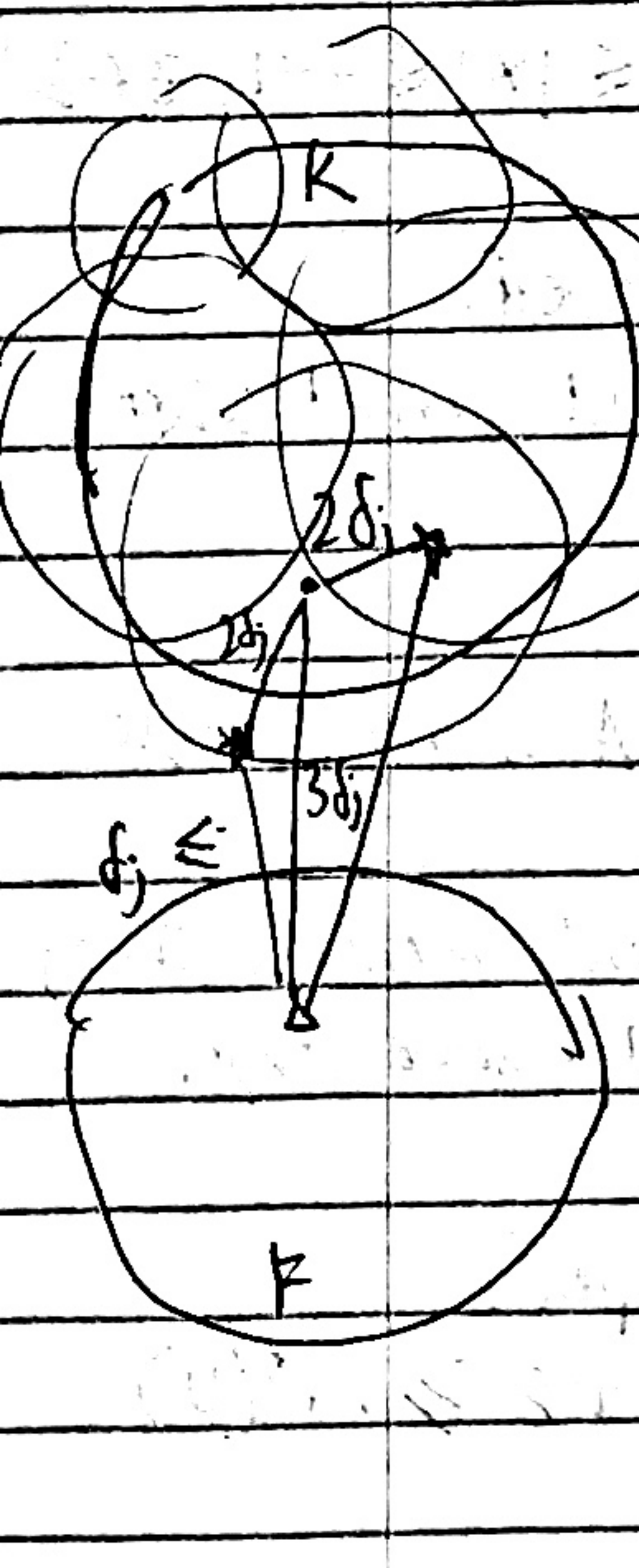
We can write $U-F$ as a countable union of "almost disjoint" cubes. Here, the cubes described are closed. We can do similarly to our proof in Lemma 1. Consider closed cubes $[0,1]^n, [0,\frac{1}{2}]^n, \dots [0,\frac{1}{2^k}]^n$ and their translations by integer multiples of their sides.

$\forall x \in U-F$, since $U-F$ open, $\exists r$ s.t. $B_r(x) \subset U-F$ and since $\exists k$ s.t. $r > \frac{1}{2^k}$,

$x \in$ a closed cube of size $\frac{1}{2^k}$.

Hence $U-F = \bigcup_{j=1}^{\infty} Q_j$ where Q_j are the countable closed cubes.

For a fixed N, the finite union $K = \bigcup_{j=1}^{N} Q_j$ is compact. (Since finite union of closed sets are closed and clearly bounded). Therefore, by another lemma, $d(K,F) > 0$.



<u>Lemma</u>: If F closed and K compact, and K, F disjoint, then $d(F,K) > 0$

<u>Proof</u>: Since F closed, for each $x \in K$, $\exists \delta_x > 0$ s.t. $d(x,F) > 3\delta_x$. (Suppose otherwise, that ~~it~~ does not exist such δ_x. Then $\forall n \in \mathbb{N}$, $d(x,F) < \frac{1}{n}$. Then we can find a sequence of points $(f_i) \in F$ ~~s.t.~~. By Bolzano Weierstrass, we can find a convergent subsequence and thus $d(x,f) = 0$ where f is the limit of subsequence. Since f closed, this implies $x \in F$. But $x \in K$ contradicting the fact that F, K disjoint)

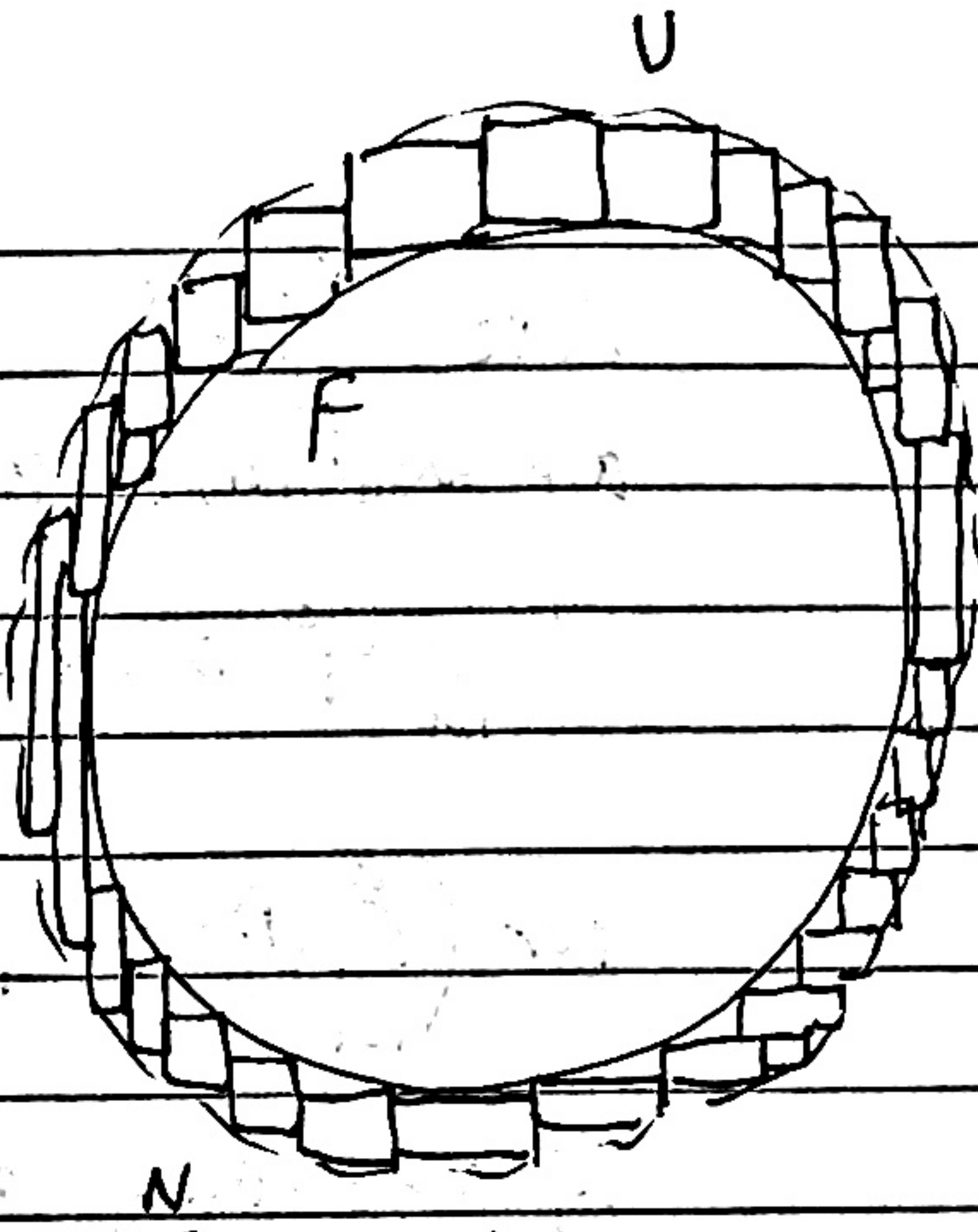
Note $\bigcup_{x \in K} B_{2\delta_x}(x)$ covers K and K compact $\Rightarrow$ can find a subcover by Heine-Borel theorem, denoted as $\bigcup_{j=1}^{N} B_{2\delta_j}(x_j)$. Pick $\delta = \min(\delta_1, \dots, \delta_N)$. Then for any $x \in K$, $y \in F$, since

$\bigcup_{j=1}^{N} B_{2\delta_j}(x_j)$ is a cover of K, $\exists j$ s.t. $x \in B_{2\delta_j}(x_j)$

△-inequality

So $|y-x| + |x-x_j| \geq |y-x_j|$

$\Rightarrow |y-x| \geq |y-x_j| - |x-x_j| \geq 3\delta_j - 2\delta_j = \delta_j \geq \delta$



$\therefore \exists \delta > 0$ s.t. $|y-x| \geq \delta \ \forall y \in F, x \in K$

$\Rightarrow d(F,K) > 0$.

Now, note that $(K \cup F) \subset U$. Hence by Monotonicity,

$$M^*(K \cup F) \leq m^*(U)$$

By lemma 0, since $\text{dist}(K,F) > 0$

$$m^*(K \cup F) = m^*(K) + m^*(F) \geq m^*(K) + \sum_{j=1}^{N} m^*(Q_j)$$

$$\text{Hence } m^*(K \cup F) = m^*(K) + m^*(F) = m^*(K) + \sum_{j=1}^{N} m^*(Q_j) \leq m^*(U)$$

$$\Rightarrow \sum_{j=1}^{N} m^*(Q_j) \leq m^*(U) - m^*(K) \leq \varepsilon$$ (as obtained in the first few steps)

Since this holds true $\forall n$, taking $N \to \infty$.

$$\lim_{N\to\infty} \sum_{j=1}^{N} m^*(Q_j) = \sum_{j=1}^{\infty} m^*(Q_j) \leq m^*(U) - m^*(K) \leq \varepsilon$$

By sub-additivity, $m^*(U-F) = m^*\left(\bigcup_{j=1}^{\infty} Q_j\right) \leq \sum_{j=1}^{\infty} m^*(Q_j) \leq \varepsilon$

$\Rightarrow$ $\boxed{F \text{ is measurable}}$ $\Rightarrow$ Every compact set is measurable $\Rightarrow$ $\boxed{\text{Every closed subset } A \subset \mathbb{R}^n \text{ is measurable.}}$

So key idea is to decompose an open set into union of "almost disjoint" cubes, then by considering finite union first, obtain equality for outer measure calculations (lemma 0). Taking limit is ok since for "almost disjoint" cubes, it can be changed to sum.

lemma 4. Since E is measurable $\Rightarrow \forall \varepsilon > 0, \exists U$ st. $m^*(U \setminus E) < \varepsilon$. ($E \subset U$)
Hence, $\forall N > 0, \exists U_N$ st. $E \subset U_N$ and $m^*(U_N \setminus E) < \frac{1}{N}$ $(N \in \mathbb{Z}^+)$

Note U_N^c is closed since U_N is open.
Also note $\boxed{E^c = \left(\bigcup_{i=1}^{\infty} U_i^c\right) \cup \left(\left(\bigcap_{i=1}^{\infty} U_i\right) \setminus E\right)}$

This is because $\forall x \notin \bigcup_{i=1}^{\infty} U_i^c$,
$x \in \bigcap_{i=1}^{\infty} U_i$
$\Rightarrow x \in \left(\bigcap_{i=1}^{\infty} U_i\right) \cap E^c \Rightarrow x \in \left(\bigcap_{i=1}^{\infty} U_i\right) \setminus E$

Note $\bigcup_{i=1}^{\infty} V_i^c$ is measurable since V_i^c is closed and by lemma 3 all closed subsets are measurable. By lemma 2, a countable collection of measurable set is measurable
$\Rightarrow \bigcup_{i=1}^{\infty} V_i^c$ is measurable

For $\left(\bigcap_{i=1}^{\infty} V_i\right)\setminus E$, $\left(\bigcap_{i=1}^{\infty} V_i\right)\setminus E = \bigcap_{i=1}^{\infty}(V_i\setminus E) \subset V_i\setminus E \quad \forall i \in \mathbb{Z}^+$

By monotonicity of outer measure, $m^*\left(\bigcap_{i=1}^{\infty}(V_i\setminus E)\right) \leq m^*(V_i\setminus E) < \frac{1}{i} \quad \forall i \in \mathbb{Z}^+$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, $m^*\left(\bigcap_{i=1}^{\infty}(V_i\setminus E)\right) = 0$

$\Rightarrow \boxed{m^*\left(\left(\bigcap_{i=1}^{\infty} V_i\right)\setminus E\right) = 0}$.

Since $m^*\left(\left(\bigcap_{i=1}^{\infty} V_i\right)\setminus E\right) = 0$, it can be covered by a union of open boxes with volume arbitrarily small

Hence $\exists$ open set U' (namely the boxes) that contain $\bigcap_{i=1}^{\infty} V_i\setminus E$ with
$m^*\left(U'\setminus\left(\bigcap_{i=1}^{\infty} V_i\setminus E\right)\right) \leq m^*(U') = 0. < \varepsilon$

Hence $\left(\bigcap_{i=1}^{\infty} V_i\right)\setminus E$ is measurable.

Since E^c is the union of measurable subsets, E^c is measurable.

$\boxed{\therefore\ E \text{ is measurable} \Rightarrow E^c \text{ is measurable.}}$