01/27/2022

MATH 105 Discussion 4 (01/27)

Definition A subset $E \subset \mathbb{R}^n$ is measurable if $\forall \varepsilon > 0$, $\exists$ open set U s.t. $U \supset E$ and $m^*(U \setminus E) < \varepsilon$.

Lemma 7.4.4 (a) HW2 will prove them (HARD)

(b) E is measurable $\Rightarrow$ $\forall \varepsilon > 0$, $\exists$ open set U s.t. $E \subset U$ and $m^*(U \setminus E) < \varepsilon$.
Then, let $\varepsilon > 0$. For set $x + E$, consider the open cover $U + x$ where the U corresponds to the same U for E with ε.

Then $E + x \subset U + x$ and $m^*(U+x \setminus E+x) = m^*(U \setminus E) < \varepsilon$
(translational invariance of outer measure)

$\Rightarrow$ $\boxed{E+x \text{ measurable.}}$

(c) Suppose E_1 and E_2 measurable. Want to show $E_1 \cup E_2$ measurable.
Let $\varepsilon > 0$.
Since E_1, E_2 measurable, $\exists$ open set U_1, U_2 s.t. $U_1 \supset E_1$, $U_2 \supset E_2$ and $m^*(U_1 \setminus E_1) < \frac{\varepsilon}{2}$ and $m^*(U_2 \setminus E_2) < \frac{\varepsilon}{2}$.

Consider $U_1 \cup U_2$, which is also open. It contains $E_1 \cup E_2$.
Let $x \in (U_1 \cup U_2) \setminus (E_1 \cup E_2)$
Since $x \in (U_1 \cup U_2) \setminus (E_1 \cup E_2)$, $x \in U_1 \cup U_2$.
If $x \in U_1$, then $x \in U_1 \setminus E_1$
If $x \in U_2$, then $x \in U_2 \setminus E_2$.
Either way, $x \in U_1 \setminus E_1$ or $U_2 \setminus E_2$

$\Rightarrow$ $(U_1 \cup U_2) \setminus (E_1 \cup E_2) \subset (U_1 \setminus E_1) \cup (U_2 \setminus E_2)$

By subadditivity of outer measure, and monotonicity

$$m^*((U_1 \cup U_2) \setminus (E_1 \cup E_2)) \leq m^*(U_1 \setminus E_1) + m^*(U_2 \setminus E_2) < \underbrace{\frac{\varepsilon}{2} + \frac{\varepsilon}{2}}_{= \varepsilon}$$

$\Rightarrow$ Since ε is arbitrarily chosen, $\boxed{E_1 \cup E_2 \text{ is measurable}}$

7.4.4 (d) Now, want to show $E_1 \cap E_2$ is measurable.
Continued

Again, let $\varepsilon > 0$.

Let V_1 and V_2 be two covers of E_1 and E_2 s.t.

$m^*(V_1 \setminus E_1) < \frac{\varepsilon}{2}$ and $m^*(V_2 \setminus E_2) < \frac{\varepsilon}{2}$. (possible since both measurable)

Consider $V_1 \cap V_2$ (which is still open)

I claim it covers $E_1 \cap E_2$

Let $x \in E_1 \cap E_2$. $\Rightarrow x \in E_1$ and $x \in E_2$

$\Rightarrow x \in V_1$ and $x \in V_2$ (since $E_1 \subset V_1$ and $E_2 \subset V_2$)

$\Rightarrow x \in V_1 \cap V_2$.

$\therefore (V_1 \cap V_2) \supset (E_1 \cap E_2)$.

$m^*((V_1 \cap V_2) \setminus (E_1 \cap E_2))$.

Let $x \in (V_1 \cap V_2) \setminus (E_1 \cap E_2)$

Then $x \in V_1$ and $x \in V_2$ but $\underbrace{x \notin E_1 \text{ or } x \notin E_2}_{\text{one of them must hold true}}$.

Suppose $x \notin E_1$, then $x \in V_1 \setminus E_1$.

$x \notin E_2$, then $x \in V_2 \setminus E_2$

Either way $x \in (V_1 \setminus E_1)$ or $x \in (V_2 \setminus E_2)$

Hence by monotonicity and subadditivity, since

$(V_1 \cap V_2) \setminus (E_1 \cap E_2) \subset (V_1 \setminus E_1) \cup (V_2 \setminus E_2)$

$$m^*((V_1 \cap V_2) \setminus (E_1 \cap E_2)) \leq m^*(V_1 \setminus E_1) + m^*(V_2 \setminus E_2)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since ε arbitrarily chosen, $\boxed{E_1 \cap E_2 \text{ is measurable}}$

(e) An open box is clearly measurable. Just take $V =$ box itself.
For a closed box, set $\delta \ll 1$. Let $SA =$ sum of all faces of closed box with δ increment to the sides (i.e. surface area of box with $\prod (a_i - \delta, b_i + \delta)$). Then consider an open box with marginally larger sides. Take $\delta \to 0$. ~~ε can approach 0.~~