8.3.2.

(a)

If $c \geq 0$ then

$$\begin{aligned} c\int_\Omega f &= c\left(\int_\Omega f^+ - \int_\Omega f^-\right) \\ &= c\int_\Omega f^+ - c\int_\Omega f^- \\ &= \int_\Omega cf^+ - \int_\Omega cf^- \\ &= \int_\Omega (cf)^+ - \int_\Omega (cf)^- \\ &= \int_\Omega cf \end{aligned}$$

If $c \leq 0$ then

$$\begin{aligned} c\int_\Omega f &= c\left(\int_\Omega f^+ - \int_\Omega f^-\right) \\ &= c\int_\Omega f^+ - c\int_\Omega f^- \\ &= -\int_\Omega -cf^+ + \int_\Omega -cf^- \\ &= -\int_\Omega (-cf)^+ + \int_\Omega (-cf)^- \\ &= -\int_\Omega (cf)^- + \int_\Omega (cf)^+ \\ &= \int_\Omega cf \end{aligned}$$

(b)

First, note that if $h : \Omega \to \mathbb{R}$ is non-negative and measurable,
and $\Omega = A \sqcup B$, then

$$\begin{aligned} &\int_\Omega h \\ &= \int_\Omega (\chi_A h + \chi_B h) \\ &= \int_\Omega \chi_A h + \int_\Omega \chi_B h \\ &= \int_A h + \int_B h \end{aligned}$$

by additivity.

Second, define:

$$\begin{aligned} A_f &= f^{-1}([0, +\infty)) \\ N_f &= f^{-1}((-\infty, 0)) \\ A_g &= g^{-1}([0, +\infty)) \\ N_g &= g^{-1}((-\infty, 0)) \\ F &= \{\omega \in \Omega \mid |f(\omega)| \geq |g(\omega)|\} \\ G &= \{\omega \in \Omega \mid |f(\omega)| < |g(\omega)|\} \end{aligned}$$

Third, note that

$$\begin{aligned} \Omega &= A_f \sqcup N_f \\ &= A_g \sqcup N_g \\ &= F \sqcup G \end{aligned}$$

Fourth, note that for any absolutely integrable function $h : \Omega \to \mathbb{R}$ and any $S \subseteq \Omega$:

$$h^{-1}((0, +\infty)) \subseteq S \subseteq h^{-1}([0, +\infty)) \implies \int_\Omega h^+ = \int_S h \tag{1}$$

$$h^{-1}((-\infty, 0)) \subseteq S \subseteq h^{-1}((-\infty, 0]) \implies \int_\Omega h^- = \int_S -h \tag{2}$$

Each integrand is non-negative.

Fifth, in the above (fourth), let $h = f + g$.

- If $S = (A_f \cup G) \cap (A_g \cup F)$, then (1) holds.
- If $S = (N_f \cup G) \cap (N_g \cup F)$, then (2) holds.

What remains is a large number of integrals.

$$
\begin{aligned}
\int_\Omega (f+g) &= \int_\Omega (f+g)^+ - \int_\Omega (f+g)^- \\
&= \int_{(A_f \cup G)\cap(A_g \cup F)} (f+g) \quad - \quad \int_{(N_f \cup G)\cap(N_g \cup F)} -(f+g) \\
&= \left(\int_{A_f \cap A_g} (f+g) + \int_{A_f \cap N_g \cap F} (f-(-g)) + \int_{A_g \cap N_f \cap G} (g-(-f)) \right) \\
&\quad - \left(\int_{N_f \cup N_g} ((-f)+(-g)) + \int_{N_f \cap A_g \cap F} ((-f)-g) + \int_{N_g \cap A_f \cap G} ((-g)-f) \right)
\end{aligned}
$$

Note that if $a, b : \Omega \to \mathbb{R}$ are non-negative functions with $a \geq b$, then

$$\int a = \int (a-b) + \int b$$

$$\int (a-b) = \int a - \int b$$

So our 6 integrals split into these 12:

$$
\begin{aligned}
&\left(\int_{A_f \cap A_g} f + \int_{A_f \cap A_g} g \right. \\
&+ \int_{A_f \cap N_g \cap F} f - \int_{A_f \cap N_g \cap F} (-g) \\
&\left. + \int_{A_g \cap N_f \cap G} g - \int_{A_g \cap N_f \cap G} (-f) \right) \\
-&\left(\int_{N_f \cup N_g} (-f) + \int_{N_f \cup N_g} (-g) \right. \\
&+ \int_{N_f \cap A_g \cap F} (-f) - \int_{N_f \cap A_g \cap F} g \\
&\left. + \int_{N_g \cap A_f \cap G} (-g) - \int_{N_g \cap A_f \cap G} f \right)
\end{aligned}
$$

which can also be written as

$$
\begin{aligned}
&\Bigg(\int_{A_f \cap A_g} f^+ + \int_{A_f \cap A_g} g^+ \\
&+ \int_{A_f \cap N_g \cap F} f^+ - \int_{A_f \cap N_g \cap F} g^- \\
&+ \int_{A_g \cap N_f \cap G} g^+ - \int_{A_g \cap N_f \cap G} f^- \Bigg) \\
-&\Bigg(\int_{N_f \cup N_g} f^- + \int_{N_f \cup N_g} g^- \\
&+ \int_{N_f \cap A_g \cap F} f^- - \int_{N_f \cap A_g \cap F} g^+ \\
&+ \int_{N_g \cap A_f \cap G} g^- - \int_{N_g \cap A_f \cap G} f^+ \Bigg)
\end{aligned}
$$

Combining integrals that have the same integrand,
we simplify to 4 intgrals:

$$
\begin{aligned}
&\left(\int_\Omega f^+ - \int_\Omega f^- \right) \\
+&\left(\int_\Omega g^+ - \int_\Omega g^- \right)
\end{aligned}
$$

which equals

$$\int_\Omega f + \int_\Omega g$$

Recalling how this all started, we find

$$\int_\Omega (f+g) = \int_\Omega f + \int_\Omega g$$

(c)

$g - f$ is absolutely integrable and ≥ 0, hence

$$\int_\Omega (g - f) \geq 0$$

By (b),

$$\begin{aligned} &\int_\Omega g \\ = &\int_\Omega (f + (g - f)) \\ = &\int_\Omega f + \int_\Omega (g - f) \\ \geq &\int_\Omega f \end{aligned}$$

(d)

Let Z be the zero set

$$Z = \{\omega \in \Omega \mid f(\omega) \neq g(\omega)\}$$

Note that if $h : \Omega \to \mathbb{R}$ is non-negative and measurable, then

$$\begin{aligned}
&\int_\Omega h \\
&= \int_\Omega (\chi_{\Omega \setminus Z} h + \chi_Z h) \\
&= \int_{\Omega \setminus Z} h + \int_Z h \\
&= \int_{\Omega \setminus Z} h + 0 \\
&= \int_{\Omega \setminus Z} h
\end{aligned}$$

by (b).
From this we deduce

$$\begin{aligned}
&\int_\Omega f \\
&= \int_{\Omega \setminus Z} f \\
&= \int_{\Omega \setminus Z} g \\
&\int_\Omega g
\end{aligned}$$

8.3.3.

$g - f$ is absolutely integrable and ≥ 0, hence

$$\int_{\mathbb{R}} (g - f) \geq 0$$

$$\int_{\mathbb{R}} g = \int_{\mathbb{R}} f + \int_{\mathbb{R}} (g - f)$$

$$\int_{\mathbb{R}} (g - f) = \int_{\mathbb{R}} g - \int_{\mathbb{R}} f = 0$$

By Proposition 8.2.6(a), it follows that

$$g - f = 0 \quad \text{a.e.}$$

$$f = g \quad \text{a.e.}$$