Convention.
For any function $f : X \to Y$, we define

$$f_* : \mathscr{P}(X) \to \mathscr{P}(Y)$$
$$f^* : \mathscr{P}(Y) \to \mathscr{P}(X)$$

where f_* is the function which maps a set to its image under f,
and f^* is the function which maps a set to its pre-image under f.

Convention.
Throughout, Ω some subset of $\mathbb{R}^m$,
and f is some function $\Omega \to \mathbb{R}_{\geq 0}$.

Definition.
The **undergraph** of f, denoted $\mathcal{U}f$,
is the set of points in $\Omega \times \mathbb{R}_{\geq 0}$ lying below the graph of f
but on or above $\Omega \times \{0\}$.
(If $\Omega = \mathbb{R}$, then $\Omega \times \{0\}$ is the x-axis.)

$$\mathcal{U}f = \{(x, y) \in \Omega \times \mathbb{R}_{\geq 0} \mid 0 \leq y < f(x)\}$$

Definition.
f is **Pugh measurable**
if its undergraph is measurable.

Definition.
The **Pugh Lebesgue integral** of a Pugh measurable function f is

$$\int_\Omega f = m(\mathcal{U}f)$$

Definition.
f is **Pugh Lebesgue integrable**
if it is measurable
and its Lebesgue integral is finite.

Claim.
Let $\Omega = [a, b]$ for some real numbers $a \leq b$,
and let f be Riemann integrable.
Then f is Pugh Lebesgue integrable and

$$\int_{[a,b]} f = \int_a^b f(x)dx$$

Proof.

> **Convention.**
> Given a partition $P = \{x_0 < \cdots < x_n\}$ of $[a, b]$,
> we denote by $U_P(f)$ the upper sum of f on that partition,
> and by $L_P(f)$ the lower sum.

For any partition P of $[a,b]$, we have

$$\begin{aligned} U_P(f) &= \sum_{i=1}^{n} \sup(f_*([x_{i-1},x_i])) \cdot m_1([x_{i-1},x_i]) \\ &= \sum_{i=1}^{n} m_2([x_{i-1},x_i] \times [0, \sup(f_*([x_{i-1},x_i]))]) \\ &\geq m_2(\bigcup_{i=1}^{n} [x_{i-1},x_i] \times [0, \sup(f_*([x_{i-1},x_i]))]) \\ &\geq m_2^*(\mathcal{U}f) \end{aligned}$$

Our next union will be disjoint. We have

$$\begin{aligned} L_P(f) &= \sum_{i=1}^{n} \inf(f_*([x_{i-1},x_i])) \cdot m_1([x_{i-1},x_i]) \\ &= \sum_{i=1}^{n} m_2([x_{i-1},x_i) \times [0, \inf(f_*([x_{i-1},x_i]))]) \\ &= m_2(\bigsqcup_{i=1}^{n} [x_{i-1},x_i) \times [0, \inf(f_*([x_{i-1},x_i]))]) \\ &\leq m_{2*}(\mathcal{U}f) \end{aligned}$$

m_{2*} is inner measure.
Since f is Riemann integrable,

$$\begin{aligned} \int_a^b f(x)dx &= \sup\{L_P(f) \mid P \text{ a partition of } [a,b]\} \\ &\leq m_{2*}(\mathcal{U}f) \\ &\leq m_2^*(\mathcal{U}f) \\ &\leq \inf\{U_P(f) \mid P \text{ a partition of } [a,b]\} \\ &= \int_a^b f(x)dx \end{aligned}$$

Hence $\mathcal{U}f$ is measurable and

$$\int_{[a,b]} f = m_2(\mathcal{U}f) = \int_a^b f(x)dx$$

I have not proven the analogous claim for Tao's definitions.
Nonetheless, here are Tao's definitions and some facts.

Convention.
From now on,
the codomain of f is $\mathbb{R}^*_{\geq 0} = [0, +\infty]$.

Definition.
f is **Tao measurable**

if $f^*(V) \subseteq \Omega$ is measurable
for every open set $V \subseteq \mathbb{R}^*_{\geq 0}$.
(Or equivalently, if this is true for any closed subset.
Or equivalently, if this is true for any interval $[a, +\infty]$ with $a \in [0, +\infty)$.)

Definition.
A **simple function** $s : \Omega \to \mathbb{R}_{\geq 0}$
is a Tao measurable function whose image $s_*(\Omega)$ is finite.

Claim (Tao 8.1.5).
If f is measurable,
then there exists a sequence $f_1, f_2, \ldots$
of simple functions $\Omega \to \mathbb{R}_{\geq 0}$ such that for any $\omega \in \Omega$:

- $f_1(\omega) \leq f_2(\omega) \leq \cdots$
- $\sup \{f_n(\omega) \mid n \in \mathbb{N}\} = f(\omega)$

Proof by construction.
For each $n \in \mathbb{N}$, let

$$P_n = \frac{\mathbb{Z}}{2^n} \cap [0, n] = \{x_0 < \cdots < x_{k_n}\}$$

For each $n \in \mathbb{N}$ and $\omega \in \Omega$, define

$$f_n(\omega) = \begin{cases} x_{i-1} & f(\omega) \in [x_{i-1}, x_i) \\ x_n & f(\omega) \in [x_n, +\infty) \end{cases}$$

Each f_n is simple (with at most $n2^n + 1$ outputs)
and both bullet points are satisfied.

Definition.
The **Tao Lebesgue integral**
of a simple function $s : \Omega \to \mathbb{R}_{\geq 0}$ is

$$\int_\Omega s = \sum_{x \in s_*(\Omega)} x \cdot m(s_*(\{x\}))$$

Definition.
If f is measurable,
the **Tao Lebesgue integral** of f is

$$\int_\Omega f = \sup \left\{ \int_\Omega s \;\middle|\; s \text{ simple}, \, s \leq f \right\}$$

Remark.
It follows that

$$\int_\Omega f = \sup \left\{ \int_\Omega g \;\middle|\; g \text{ measurable}, \, g \leq f \right\}$$

Claim.
Let f be Tao measurable.

Then f is Pugh measurable
and its Pugh Lebesgue integral equals its Tao Lebesgue integral:

$$\int_{\Omega}^{(P)} f = \int_{\Omega}^{(T)} f$$

Proof.

Write-up in progress.