1.

(a) Special case of (b).

(b)

Consider the hyperplane

$$H = x + \langle v_1, \dots v_{n-1} \rangle$$

where $x, v_i \in \mathbb{R}^n$ and v_i are linearly independent.
(The angle brackets denote "subspace generated by".)
To determine $m^*(H)$, we let $x = 0$ WLOG since outer measure is translation invariant.
Let $\hat{x}_i$ be the i^{th} standard basis vector of $\mathbb{R}^n$.
Since $\dim H = n - 1$ (as a vector space), there's an i such that

$$\hat{x}_i \notin H$$

WLOG $i = n$.
Now consider the linear map T such that

$$v_i \mapsto \hat{x}_i$$

$$\hat{x}_n \mapsto \hat{x}_n$$

This maps a basis to a basis, so it's invertible, an isomorphism, an affine motion, and has nonzero determinant d. Furthermore, it maps $H = \langle v_1, \dots v_{n-1} \rangle$ as follows:

$$T_* : \langle v_1, \dots v_{n-1} \rangle \mapsto \langle \hat{x}_1, \dots \hat{x}_{n-1} \rangle$$

i.e. $T_* H = \mathbb{R}^{n-1} \times \{0\}$. Since T is a meseomorphism and $\mathbb{R}^{n-1} \times \{0\}$ is measurable, this shows that H is measurable. Hence

$$\begin{aligned} d \cdot m(H) &= m(T_* H) \\ &= m(\mathbb{R}^{n-1} \times \{0\}) \\ &= 0 \end{aligned}$$

$$m(H) = 0$$

The set $\{x = y\} \subseteq \mathbb{R}^2$.

Let $S = \{x = y\}$. We will prove that $m(S) = 0$ using boxes.
Pick an injection

$$\mathbb{Z} \to \mathbb{N}_0$$

$$n_i \mapsto i$$

This gives

$$S = \bigsqcup_{i=0}^{\infty} (S \cap [n_i, n_i + 1) \times \mathbb{R})$$

For each $i \in \mathbb{N}_0$, form the set

$$\mathcal{B}_i = \left\{ B_{ij} \;\middle|\; 0 \le j < 2^i \right\}$$

$$B_{ij} = [n_i + \frac{j}{2^i}, n_i + \frac{j+1}{2^i})^2$$

Lemma 1. *$\bigsqcup_i \bigsqcup_{j<2^i} B_{ij}$ covers S.*

Let $p = (x, x) \in S$. Then for some $n \in \mathbb{Z}$:

$$n \leq x < n + 1$$

Using the function $n_j \mapsto j$, we find an $i \in \mathbb{N}_0$ such that

$$n = n_i$$

Now let

$$y = (x - n) \cdot 2^i$$

There is some $j < 2^i$ such that

$$j \leq y < j + 1$$

This now gives

$$n_i + \frac{j}{2^i} \leq x < n_i + \frac{j+1}{2^i}$$

$$p = (x, x) \in [n_i + \frac{j}{2^i}, n_i + \frac{j+1}{2^i})^2 = B_{ij}$$

Since $p \in S$ was arbitrary, Lemma 1 is proven.

Now note that

$$|B_{ij}| = 2^{-2i}$$

$$\sum_{j<2^i} |B_{ij}| = 2^{-i}$$

Thus, letting the image of $n_i \mapsto i$ be contained in

$$\mathbb{N}_0 \setminus \{0, 1, \ldots, m-1\}$$

for some m, we find

$$\begin{aligned}
\sum_i \sum_{j<2^i} |B_{ij}| &= \sum_i 2^{-i} \\
&\leq \sum_{i=m}^{\infty} 2^{-i} \\
&= 2^{-(m-1)}
\end{aligned}$$

Hence, by picking a sufficiently large $m \in \mathbb{N}_0$, we cover $S = \{x = y\}$ with boxes having an arbitrarily small sum of volumes.

$$m(\{x = y\}) = 0$$

2.

Theorem 11.

Bounded $\implies$ unbounded.

Measurable $\implies$ regularity sandwich.

Let $E \subseteq \mathbb{R}^n$ be measurable. Partitioning E as

$$E = \sqcup_{i\in\mathbb{N}} E_i$$

with E_i bounded, we may take an F_σ $F_i \subseteq E_i$ for each i, with

$$m(E_i \setminus F_i) = 0$$

Letting $F = \bigsqcup_{i\in\mathbb{N}} F_i \subseteq E$, we find that F is also an F_σ and

$$\begin{aligned} m(E \setminus F) &= m(\bigsqcup_{i\in\mathbb{N}} (E_i \setminus F_i)) \\ &= \sum_{i\in\mathbb{N}} m(E_i \setminus F_i) \\ &= 0 \end{aligned}$$

Now let $(a_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers with infimum 0. For each i, let $G_i \supseteq E_i$ be a G_δ with

$$G_i = \bigcap_{j=1}^{\infty} G_{ij}$$

$$m(G_i \setminus E_i) = 0$$

Furthermore, choose the open sets G_{ij} so that

$$m(G_{ij} \setminus E) < \frac{a_j}{2^i}$$

Now let

$$G_j = \bigcup_{i=1}^{\infty} G_{ij}$$

G_j is open and

$$G_j \supseteq E$$

$$\begin{aligned} G_j \setminus E &= \bigcup_{i=1}^{\infty} G_{ij} \setminus E \\ &\subseteq \bigcup_{i=1}^{\infty} G_{ij} \setminus E_i \end{aligned}$$

$$\begin{aligned} m(G_j \setminus E) &\leq m\left(\bigcup_{i=1}^{\infty} G_{ij} \setminus E_i\right) \\ &\leq \sum_{i=1}^{\infty} m(G_{ij} \setminus E_i) \\ &< \sum_{i=1}^{\infty} \frac{a_j}{2^i} \\ &= a_j \end{aligned}$$

Now, considering the G_δ

$$G = \bigcap_{j=1}^{\infty} G_j$$

and recalling that the infimum of the a_j is 0, we find by monotonicity that

$$m(G \setminus E) = 0$$

$$\begin{aligned} m(G \setminus F) &= m(G \setminus E) + m(E \setminus F) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Regularity sandwich $\implies$ measurable.

Trivial.

Unbounded $\implies$ bounded.

Measurable $\implies$ regularity sandwich.

Let E be a bounded measurable set. Letting

$$M = \sup \{|e| \mid e \in E\}$$

we form

$$E' = \{k\hat{x}_1 \mid k \in \mathbb{N}, k > M\}$$

E' is an unbounded nullset disconnected from E.
Since $E \sqcup E'$ is unbounded, there is a regularity sandwich

$$F \subseteq E \sqcup E' \subseteq G$$

We will now prove that

$$F \cap E \subseteq E \subseteq G$$

is also a regularity sandwich.

Part 1. $m(G \setminus E) = 0$

$$\begin{aligned} m(G \setminus E) &= m(G \setminus (E \sqcup E')) + m((E \sqcup E') \setminus E) \\ &= m(G \setminus (E \sqcup E')) + mE' \\ &= m(G \setminus (E \sqcup E')) \\ &= 0 \end{aligned}$$

Part 2. *$(F \cap E)$ is an F_σ.*

Lemma. *If A, B are disconnected subsets of a topological space, and $A \sqcup B$ is closed, then A is closed.*

Let $x \in \overline{A}$.
Then x is in every closed set containing A, including $A \sqcup B$.
Hence $x \in A$ or $x \in B$.
But since A, B are disconnected, $\overline{A}$ is disjoint from B, so $x \in A$.
Since x was arbitrary, $\overline{A} \subseteq A$ and A is closed.

Write

$$F = \bigcup_{i=1}^{\infty} F_i$$

with F_i closed. Then

$$F \cap E = \bigcup_{i=1}^{\infty} F_i \cap E$$

In the above lemma, let

$$A = F_i \cap E$$

$$B = F_i \cap E'$$

A is closed, so F is an F_σ.

Part 3. $m(E \setminus (F \cap E)) = 0$

$$\begin{aligned} m(E \setminus (F \cap E)) &= m(E \setminus F) \\ &= m(E \setminus F) + 0 \\ &= m(E \setminus F) + m(E' \setminus F) \\ &= m((E \sqcup E') \setminus F) \\ &= 0 \end{aligned}$$

Regularity sandwich $\implies$ measurable.

Trivial.

Theorem 21.

Bounded $\implies$ unbounded.

Let $A \subseteq \mathbb{R}^m, B \subseteq \mathbb{R}^n$. Disjointize:

$$A = \bigsqcup_{i=1}^{\infty} A_i$$

$$A_i \text{ bounded}$$

and similar for B. Now

$$A \times B = \bigsqcup_{(i,j) \in \mathbb{N}^2} A_i \times B_j$$

$$\begin{aligned} m_{m+n}(A \times B) &= m_{m+n}\left(\bigsqcup_{(i,j) \in \mathbb{N}^2} A_i \times B_j \right) \\ &= \sum_{(i,j) \in \mathbb{N}^2} m_{m+n}(A_i \times B_j) \\ &\overset{21}{=} \sum_{(i,j) \in \mathbb{N}^2} m_m(A_i) m_n(B_j) \\ &= \left(\sum_{i \in \mathbb{N}} m_m(A_i) \right) \left(\sum_{j \in \mathbb{N}} m_n(B_j) \right) \\ &= m_{m+n} A \cdot m_{m+n} B \end{aligned}$$

Unbounded $\implies$ bounded.

Let $A \subseteq \mathbb{R}^m, B \subseteq \mathbb{R}^n$.

If A is bounded, let A' be an unbounded nullset disjoint from A (e.g. a countable set of increasingly and sufficiently far-away points). Otherwise, define $A' = \emptyset$. Similar for B.

Recall that, due to a Tao exercise on hw1,

$$m^*_{m+n}(A' \times B) \leq m^*_{m+n}(A')m^*_{m+n}(B) = 0$$

Now note that

$$(A \sqcup A') \times (B \sqcup B') = (A \times B) \sqcup (A' \times B) \sqcup (A \times B') \sqcup (A' \times B')$$

$$\begin{aligned}
m_{m+n}(A \times B) &= m_{m+n}(A \times B) + 0 + 0 + 0 \\
&= m_{m+n}(A \times B) + m_{m+n}(A' \times B) + m_{m+n}(A \times B') + m_{m+n}(A' \times B') \\
&= m_{m+n}\left(A \sqcup A' \times B \sqcup B'\right) \\
&= m_m\left(A \sqcup A'\right) m_n\left(B \sqcup B'\right) \\
&= m_m\left(A\right) m_n\left(B\right)
\end{aligned}$$

3.

Part 1. $J^*A = J^*\overline{A}$.

By monotonicity,

$$J^*A \leq J^*\overline{A}$$

Now pick $\varepsilon > 0$ and let $(B_r)_{r\in R}$ be a finite open-box covering of A with

$$\sum_{r\in R} |B_r| < J^*A + \varepsilon$$

Write

$$B_r = \prod_{i=1}^{n} (a_{ri}, b_{ri})$$
$$l_{ri} = b_{ri} - a_{ri}$$

so that $|B_r| = \prod_{i=1}^{n} l_{ri}$.

We now pick a $\gamma > 0$ and define

$$B'_r = \prod_{i=1}^{n} (a_{ri} - \frac{\gamma}{2}, b_{ri} + \frac{\gamma}{2})$$

Fact 1. *$(B'_r)_{r\in R}$ covers $\overline{A}$*

This is because

$$\overline{A} \subseteq \bigcup_{a\in A} B_{\gamma/2}(a) \subseteq \bigcup_{r\in R} B'_r$$

The volume of $|B'_r|$ is

$$|B'_r| = \prod_{i=1}^{n} (l_i + \gamma) = \sum_{I\subseteq\{1,\dots n\}} \gamma^{n-|I|} \cdot \prod_{i\in I} l_i$$

The sum is a sum of 2^n products.

Now let M be the maximum of all partial products of the side lengths of B_r:

$$M = \max \left\{ \prod_{i\in I} l_i \;\middle|\; I \subseteq \{1, \dots n\} \right\}$$

Requiring that $\gamma \leq 1$, this gives

$$\begin{aligned}
|B'_r| &= \sum_{I\subseteq\{1,\dots n\}} \gamma^{n-|I|} \cdot \prod_{i\in I} l_i \\
&= |B_r| + \sum_{\emptyset\neq I\subseteq\{1,\dots n\}} \gamma^{n-|I|} \cdot \prod_{i\in I} l_i \\
&= |B_r| + \gamma \left(\sum_{\emptyset\neq I\subseteq\{1,\dots n\}} \gamma^{n-|I|-1} \cdot \prod_{i\in I} l_i \right) \\
&\leq |B_r| + \gamma \left(\sum_{\emptyset\neq I\subseteq\{1,\dots n\}} \gamma^{n-|I|-1} M \right) \\
&\leq |B_r| + \gamma \left(\sum_{\emptyset\neq I\subseteq\{1,\dots n\}} M \right) \\
&= |B_r| + \gamma \left(2^{n-1} M \right)
\end{aligned}$$

This brings us to:
Fact 2. *By choosing a sufficiently small* γ*, we may bring* $|B'_r|$ *arbitrarily close to* $|B_r|$ *for every* $r \in R$*.*
In particular, we may obtain

$$|B'_r| \le |B_r| + \frac{\varepsilon}{|R|}$$

$$\begin{aligned} J^*\overline{A} &\le \sum_{r \in R} |B'_r| \\ &\le \left(\sum_{r \in R} |B_r| \right) + \varepsilon \\ &< J^*A + 2\varepsilon \end{aligned}$$

Since ε was aribtrary, we finally have

$$J^*\overline{A} \le J^*A$$
$$J^*A = J^*\overline{A}$$

Part 2. $J^*\overline{A} = m\overline{A}$.
Immediately, $J^*\overline{A} \ge m\overline{A}$ because finite coverings are countable.
Next, pick $\varepsilon > 0$ and let $(B_i)_{i \in \mathbb{N}}$ be a sequence of open boxes covering $\overline{A}$ such that

$$\sum_{i \in \mathbb{N}} |B_i| < m\overline{A} + \varepsilon$$

By the compactness of $\overline{A}$, there is a finite $I \subseteq \mathbb{N}$ such that $(B_i)_{i \in I}$ also covers $\overline{A}$. Hence

$$\begin{aligned} J^*\overline{A} &\le \sum_{i \in I} |B_i| \\ &\le \sum_{i \in \mathbb{N}} |B_i| \\ &< m\overline{A} + \varepsilon \end{aligned}$$

Since ε was arbitrary,

$$J^*\overline{A} \le m\overline{A}$$
$$J^*\overline{A} = m\overline{A}$$