Rudin 12.

The range of f is a torus. We might say it has "radius" b and "thickness" a. (This is a 2D surface.)

(a)

For any $(s,t) \in \mathbb{R}^2$ we have

$$(Df_1)_{(s,t)} = \begin{bmatrix} -a\sin(s)\cos(t) & -(b+a\cos(s))\sin(t) \end{bmatrix}$$

When this is $\begin{bmatrix} 0 & 0 \end{bmatrix}$, we have

$$-(b+a\cos(s))\sin(t) = 0$$

Equivalently, $0 \in \{b + a\cos(s), \sin(t)\}$.
Since $|a\cos(s)| < |b|$, we find that

$$b + a\cos(s) \neq 0$$

hence

$$\sin(t) = 0$$

We also have

$$\sin(s)\cos(t) = 0$$

Since $\sin(t) = 0$, we have $\cos(t) = 0$ and therefore

$$\sin(s) = 0$$

So the equality

$$(Df_1)_{(s,t)} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

holds only if $\sin(s), \sin(t) = 0$, and clearly the converse is true as well.
Equivalent to "$\sin(s), \sin(t) = 0$" is the following pair of equations:

$$s = k_s\pi \quad \text{for some } k_s \in \mathbb{Z}$$

$$t = k_t\pi \quad \text{for some } k_t \in \mathbb{Z}$$

When s, t are as above, we find that $f(s,t)$ can be any one of the four vectors specified by:

$$\begin{bmatrix} \pm b \pm a \\ 0 \\ 0 \end{bmatrix}$$

(b)

For any $(s,t) \in \mathbb{R}^2$ we have

$$(Df_3)_{(s,t)} = \begin{bmatrix} -a\cos(s) & 0 \end{bmatrix}$$

For this to be $\begin{bmatrix} 0 & 0 \end{bmatrix}$ is equivalent to

$$-a\cos(s) = 0$$

$$s = \frac{\pi}{2} + k \quad \text{for some } k \in \mathbb{Z}$$

So the set in question is the image of all $(s,t) \in \mathbb{R}^2$ such that s satisfies the above equation. It is the set

$$\left\{ f(s,t) \;\middle|\; s = \frac{\pi}{2} + k \;\text{ for some } k \in \mathbb{Z} \right\}$$
$$= \left\{ \begin{bmatrix} b\cos(t) \\ b\sin(t) \\ \pm a \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

(c)

Clearly, for any $(s,t) \in \mathbb{R}^2$,

$$-b - a \leq f(s,t) \leq b + a$$

So the points

$$\begin{bmatrix} -b-a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b+a \\ 0 \\ 0 \end{bmatrix}$$

found in part (a) correspond to a local minimum and a local maximum of f_1, respectively. The other two points correspond to

$$f_1(s,t) = \pm(b-a)$$

and are not local extrema of f_1.

Rudin 13.

Let $t \in \mathbb{R}$. We may write, uniquely,

$$\mathbf{f}'(t) = \mathbf{a} + k\mathbf{f}(t)$$

with $\mathbf{a} \cdot \mathbf{f}(t) = 0$. Fix $\mathbf{a}$ and k as above.

Suppose that

$$\mathbf{f}'(t) \cdot \mathbf{f}(t) \neq 0$$

For any $h \in \mathbb{R}$, we have

$$\mathbf{f}(t+h) = \mathbf{f}(t) + h\mathbf{f}'(t) + R(h)$$

$$\mathbf{f}(t+h) = \mathbf{f}(t) + h(\mathbf{a} + k\mathbf{f}(t)) + R(h)$$

$$\mathbf{f}(t+h) = (1+hk)\mathbf{f}(t) + h\mathbf{a} + R(h)$$

$$(1+hk)\mathbf{f}(t) + h\mathbf{a} = \mathbf{f}(t+h) - R(h)$$

Since $\mathbf{a} \cdot \mathbf{f}(t) = 0$, we have

$$|1+hk| = |1+hk||\mathbf{f}(t)|_{\mathbb{R}^3} \leq |(1+hk)\mathbf{f}(t) + h\mathbf{a}|_{\mathbb{R}^3}$$

We also have

$$|\mathbf{f}(t+h) - R(h)|_{\mathbb{R}^3} \leq |\mathbf{f}(t+h)|_{\mathbb{R}^3} + |R(h)|_{\mathbb{R}^3}$$

This gives

$$|1+hk| \leq |\mathbf{f}(t+h)|_{\mathbb{R}^3} + |R(h)|_{\mathbb{R}^3}$$

Now, choose an h sufficiently small and with appropriate sign, so that

$$|R(h)|_{\mathbb{R}^3} < |hk|$$

$$hk > 0$$

This gives

$$1 + hk = |1+hk| \leq |\mathbf{f}(t+h)|_{\mathbb{R}^3} + |R(h)|_{\mathbb{R}^3}$$

$$1 < 1 + (hk - |R(h)|_{\mathbb{R}^3}) \leq |\mathbf{f}(t+h)|_{\mathbb{R}^3} +$$

So

$$|\mathbf{f}(t+h)|_{\mathbb{R}^3} \neq 1$$

which we know to be false. From this contradiction, we determine that

$$\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$$

Rudin 19.

Pugh 14.

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, define

$$M = \max |Te_j|_{\mathbb{R}^n}$$

and note that

$$M \le ||T|| \le nM$$

The first inequality is immediate and the second was shown in a previous homework.

(a)

Not open.

We'll prove this for $n = 2$; the general case is similar. We'll show that

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is not in the interior of the set. Since it is also a member of the set, this will prove that the set is not open.

Let $\varepsilon > 0$. For any $c \in \mathbb{R}$ with

$$0 < |c| < \frac{\varepsilon}{n}$$

we let

$$T = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}$$

This gives $M = |c|$ and

$$||T|| \le n|c| < \varepsilon$$

Yet the matrix

$$I + T = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$$

is not diagonalizable. So there are non-diagonalizable matrices arbitrarily close to I, hence it is not in the interior of the set, so the set is not open.

(b)

Not closed.

We'll prove this for $n = 2$; the general case is similar. Consider a sequence of matrices

$$T_k = \begin{bmatrix} 1 & 1 \\ c_k & 1 \end{bmatrix}$$

such that $c_k \to 0$ as $k \to \infty$, but each c_k is nonzero.

For any $k \in \mathbb{N}$, T_k is diagonalizable, but the limit of the T_k is clearly

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

which is not diagonalizable.

(c)

Not dense.

We'll prove this for $n = 2$; the general case is probably similar. Consider the non-diagonalizable matrix

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We will show every matrix sufficiently close to T has non-real eigenvalues, and hence is also non-diagonalizable. This will show that T is not in the closure of our set, so the set is not dense.
Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be some matrix. We have

$$T + A = \begin{bmatrix} a & -1+b \\ 1+c & d \end{bmatrix}$$

Wolfram Alpha reveals that the eigenvalues of $T + A$ are

$$\lambda_1 = \tfrac{1}{2}(\pm\sqrt{a^2 - 2ab + 4bc + 4b - 4c + b^2 - 4} + a + b)$$

When A is small, M will also be small, so a, b, c, d will all be small, hence the radicand

$$a^2 - 2ab + 4bc + 4b - 4c + b^2 - 4$$

will be close to -4. In particular, it will be negative, so the eigenvalues will be non-real.
Hence, all the matrices within a certain small radius of T are non-diagonalizable (as real matrices).

Pugh 24.